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Modeling the Term Structure of Exchange Rate Expectations

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Abstract

Modeling the Term Structure of Exchange Rate Expectations

In international finance, exchange rates and exchange rate expectations represent one of the most important variables. However, the expectation of exchange rates is typically taken only with respect to a certain point of time in the future, neglecting the relevance of other dates and without checking for the (non trivial) consistence of these expectations when considering the entire term structure. Building on the well known model of Cox, Ingersoll and Ross (1985b) of the term structure of interest rates and on uncovered interest parity, we develop a model of the term structure of exchange rate expectations. We show that there are several rational methods of building expectations which are not mutually consistent. Finally, we generalize the reaction function of the spot rate on changes of the basic economic variables such as the interest rate. A sensitivity analysis shows that standard results from the simple one point expectations (e.g. overshooting) remain valid and are consistent in expectations for most cases. The general approach additionally allows to analyze the magnitude of these reactions and thus better explains the stylized facts.

JEL–Classification: F31, D84, E43

Keywords: exchange rates; expectation; term structure;

1 Motivation

Economic situations, investment decisions and even the cost of travelling are influenced by the cost of foreign currency. Beside the spot exchange rate the expectation of its future value has important implications on economic decisions. The pressure of an appreciation and depreciation, respectively, on the exchange rate influences the movement of capital and investment. Investments and their yields in the home or foreign country depend critically on the expected exchange rate in the future. The exchange rate risk determines the risk premium for investments in a foreign country and has led to many financial instruments used to hedge against this risk. Furthermore, the costs and, as a result, the competitiveness of exported and imported goods are fundamentally dependent on the exchange rate. In addition to that, the exchange rate politics of a central bank are influenced by the expectations of the future values of the exchange rate. These examples and further considerations emphasize the importance of the exchange rates and the expectation of their future development. Considering all these facts it is not surprising that researchers spend a great effort to explain the mechanism of determination of the spot exchange rate, the future expectation and their characteristics. Two main strands of economic literature on exchange rates can be differentiated. The first basically deals with exchange rate crises, their causes, consequences etc. The second, the one this paper aims to contribute to, describes the influence of exchange rates on the economy and on economic decisions in general without distinct reference to crisis situations as described in the examples noted above.

In this paper, we present possibilities of evolving paths of expected exchange rates and investigate their particular structure and behavior. The model of Cox et al. (1985b) serves as a cornerstone for modelling the term structure and its implications. As interest rates interact with exchange rates and vice versa, an approach of investigating the expectations needs to take the term structure into consideration. We hold the view that the model of Cox et al. (1985b) is perfectly suited for this task. Contrary to other models, the expectation of the exchange rate is not only investigated at one particular future time, but the whole path of expected values is examined. This approach takes into consideration that, similar to the term structure, there exist investors with different preferences regarding the length and other characteristics of investment opportunities. Consequently, various expectations at different future times are formed.

First of all, the underlying model and the main results of the papers of Cox, Ingersoll, and Ross (1985a,b) are introduced in section 2. In section 3, expectations of future values of the exchanges rates are evaluated. It is important to take into consideration that there are several possibilities of forming expectations. In the sections 3.1 and 3.2 two different approaches of calculating the expected exchange rates and their depreciation rates are presented. The results and possible similarities and differences of both approaches, respectively, are compared in section 3.3. An explicit form of the notation introduced in section 2 and its characteristics are presented in section 4. Finally, we generalize the reaction function of the spot rate on changes of the basic economic variables like interest rate. A sensitivity analysis shows that standard results (like overshooting) from the simple one point expectations remain valid and are consistent in expectations in most cases. The general approach additionally allows for an analysis of the magnitude of these reactions and may, therefore, represent an extension and a more realistic approach. The appendix contains a list of the used notation, some calculations on the formal representation of a weak form of the liquidity preference hypothesis used to limit the variations of the parameters in the sensitivity analysis, and a sensitivity analysis of the expected total one period return of a bond within the Cox et al. (1985b) framework.

2 The underlying model

In our approach we analyze the structure exchange rate expectations in a two country (country X and country Y) model without capital controls. Note that all variables used to describe the economy of country Y are indicated by *. When describing the exchange rate or the expectation of the future exchange rate, the price quotation is used from country X's point of view. The investigations will be kept as general as possible such that the results may be specialized to investigate particular problems. In order to describe the structure and the characteristics of the expected exchange rates of those two currencies, we applied the model of Cox et al. (1985b) for modelling the term structure to each of the two countries. It is possible to examine various situations as we only restricted the variables describing the economies to certain reasonable intervals, but not to certain values. There is no trading of goods between the countries. As a result, the theory of the purchasing power parity is neglected in our approach. The only way the expectations of the future exchange rate appreciation and depreciation, respectively, are evolved is that of different investment opportunities in bonds. That is, the investors observe the term structure of their own country and that of the foreign country. Differences in yields-to-maturity are seen as the reason for changes in the exchange rate in the future to ensure that the uncovered interest rate parity is valid.

Cox, Ingersoll and Ross (1985a) develop a general equilibrium asset pricing model for use in applied research. The main result is the endogenously determined price of any asset in terms of the underlying variables in the economy. It follows from the solution of a partial differential equation, which needs to be satisfied by the asset prices. Several assumptions are made to describe a simple, however easily extended, model of the economy. In Cox et al. (1985b), the authors use the intertemporal general equilibrium asset pricing model mentioned above to study the term structure of interest rates. The underlying general equilibrium model is modified and specialized to suit the needs of the described problem. The term structure of interest rates describes the relationship among the yields on default-free securities that differ only in their term to maturity. Therefore, it embodies the anticipations of the members of the market for future events. In this model, the determinants of the term premiums are studied and how changes of variables will effect the term structure. However, it also considers ideas of several, well known, approaches to the determination of the term structure, like the *expectations hypothesis*, the *liquidity hypothesis*, and the *market segmentation hypothesis*.

For our purposes the interest rate dynamics are of major importance. The dynamics correspond to a continuous time first-order autoregressive process. It is mean reverting if $\kappa > 0$ and $\theta > 0$, i.e. the randomly moving interest rate is elastically pulled toward θ. The variable θ is the long-term value and κ is the speed of adjustment. This structure leads to an interest rate behavior, which is empirically relevant. Negative spot rates are precluded and if the spot rate reaches zero, it can subsequently become positive. The absolute variance increases with an increase in r.

Furthermore, Cox et al. (1985b) give certain economically reasonable conditions on the variables, which can be written as follows:

$$\begin{aligned} \kappa\theta &\geq 0 && (1) \\ \sigma^2 &> 0 && (2) \\ \lambda &< 0 && (3) \\ \kappa &> 0 && (4) \\ \theta &> 0 && (5) \\ \kappa &> |\lambda| && (6) \end{aligned}$$

The condition on λ ensures positive premiums, since $P_r < 0$.

The bond prices are calculated as follows:

$$P(r,t,T) = A(t,T)e^{-B(t,T)r}. \tag{7}$$

Using the notation of Brown and Dybwig (1986), i.e. $\phi_1 = [(\kappa+\lambda)^2 + 2\sigma^2]^{\frac{1}{2}}$, $\phi_2 = \frac{\kappa+\lambda+\phi_1}{2}$, and $\phi_3 = \frac{2\kappa\theta}{\sigma^2}$, $A(t,T)$ and $B(t,T)$ can be written as:

$$A(t,T) \equiv \left(\frac{\phi_1 e^{\phi_2(T-t)}}{\phi_2(e^{\phi_1(T-t)} - 1) + \phi_1} \right)^{\phi_3} \tag{8}$$

$$B(t,T) \equiv \frac{e^{\phi_1(T-t)} - 1}{\phi_2(e^{\phi_1(T-t)} - 1) + \phi_1}. \tag{9}$$

Bonds are commonly quoted in terms of yields rather than prices. For the discount bonds we are now considering the yield-to-maturity $R(r,t,T)$ is defined by:

$$\begin{aligned} P(r,t,T) &= e^{-(T-t)R(r,t,T)} \\ R(r,t,T) &= \frac{rB(t,T) - ln(A(t,T))}{(T-t)}. \end{aligned} \tag{10}$$

As maturity nears, the yield-to-maturity approaches the current interest rate independently of any of the parameters. As we consider longer and longer maturities, the yield approaches a limit which is independent of the current interest rate:

$$Rl = R(r,t,\infty) = \frac{2\kappa\theta}{\gamma + \kappa + \lambda}. \tag{11}$$

The long-term yield (11) and

$$Rg = \frac{\kappa\theta}{\kappa + \lambda} \tag{12}$$

determine the appearance of the term structure.

Additionally, a liquidity preference is assumed. Several possibilities exist to define what exactly a liquidity preference means. First of all, one could say that the return of bonds with longer maturities is always higher than the return from investing in bonds with shorter time-to-maturity repeatedly, that is, the inequality is assumed to exist at any time. However, we only assume that the return of bonds with a long time-to-maturity is only higher than the return from investing in bonds with shorter time-to-maturity repeatedly if the maturity date is not in the relatively near future, but in the sufficiently distant future. Hence, the model is less restricted. The liquidity preference, as defined above, can be fulfilled by assuming $R(r,t,\infty) > \theta$. Using various results of Cox et al. (1985b) stated above and assuming that the expected future one period total return at time s can be approximated by $e^{E(r(s)|r(t))B(s,s-t)-ln(A(s,s-t))}-1$, where $E(r(s)|r(t))$ is calculated as in (40), the assumption of a liquidity preference for bonds with a maturity date in the sufficiently distant future can be written as:[1]

$$e^{r(t)\Delta} e^{E(r(t+\Delta)|r(t))\Delta} \cdots e^{E(r(T)|r(t))\Delta} < e^{R(r,t,T)(T-t)}. \tag{13}$$

[1]For further information refer to Appendix 7.2.

Analogous to that, a liquidity preference in country Y can be ensured by assuming $R^*(r^*, t, \infty) > \theta^*$.

Another approach could have been to extend the model of Cox et al. (1985b) to be an open economy, e.g. by introducing state variables describing the foreign country. We did not pursue that approach. However, we would like to point out that there are a few approaches of extending the model, or similar models, to be an open economy. For example, refer to Pavlova and Rigobon (2003).

3 The Expected Exchange and Depreciation Rates

In this section, expectations of the future exchange rates and expectations of the future depreciation rates are formed. There are several possibilities of calculating an expectation of the exchange rate at a future point of time as well as there exist various ways of evolving the particular expected depreciation rates.

In the following, the current time t is chosen to be *zero* and the current exchange rate $\mathcal{E}_0$ is assumed to be equal to *one*. This standardization does not represent a restriction to the results presented in the latter part. Additionally, we chose the length of a period to be 0.01. We have to point out, that the choice of this particular value is absolutely arbitrary.

3.1 Approach Number One

The first approach presented is using the term structure evolved by the model of Cox et al. (1985b). Using the uncovered interest rate parity one can infer the entire path of expected future values of the spot exchange rate given the observed term structure of domestic and foreign interest rates and the spot exchange rate. As the agents expect the assumptions of the interest rate parity to be fulfilled eventually, the expectation of the future spot exchange rate at any future time s is formed according to the parity condition. As the model assumes continuous payments of interest, it is important to distinguish between the yield-to-maturity

$R(r,t,T)$ and the total return $R(r,t,T)^{TR}$, which can be written as:

$$R(r,t,T)^{TR} = e^{R(r,t,T)\cdot(T-t)} - 1, \tag{14}$$

where $T-t$ stands for the time-to-maturity. It can be easily shown that (14) is equivalent to

$$R(r,t,T)^{TR} = \frac{1}{P(r,t,T)} - 1. \tag{15}$$

As a result, the expectation of the exchange rate at time T can be written as follows:[2]

$$\begin{aligned} \mathcal{E}_T^e &= \frac{1+R(r,t,T)^{TR}}{1+R^*(r^*,t,T)^{TR}}\mathcal{E}_t & (16)\\ &= \frac{P^*(r^*,t,T)}{P(r,t,T)}\mathcal{E}_t. & (17) \end{aligned}$$

The expectation regarding the exchange rate at any future time T can be used to calculate the expected depreciation rate. Taking the values of the expected exchange rates at time s and $s+\Delta$ the expected depreciation rate can be written as follows:

$$\mathbb{E}^1\left(d_s\right) = \left(\frac{\mathcal{E}_{s+\Delta}^e - \mathcal{E}_s^e}{\mathcal{E}_s^e}\right)_1. \tag{18}$$

The index $(\cdot)_1$ is introduced for later purposes. Positive values of (18) represent an expected depreciation of the exchange rate. Negative values indicate an appreciation.

3.2 Approach Number Two

In comparison with (18), the expected total one period returns ETR (Δ) and the ETR $(\Delta)^*$ can be used to calculate another expectation of the one period depreciation rate.

[2] We implemented the correct definition of the uncovered interest rate parity instead of the approximation, because the approximation errors may not be ignored if longer maturities are investigated.

By the uncovered interest rate parity condition the expected rate of exchange rate depreciation is just equal to the relevant interest rate differential. That is, the expected exchange rate depreciation is equal to the interest differential on financial assets in the relevant currencies with the same maturity and identical risk characteristics. We now assume that UIP holds in expectation, too. As a result, using the assumption of investing one unit of currency of country X in either country X or in country Y we use the following equation:

$$1 + \mathbb{E}_t\left(R(r(s), s, s+\Delta)^{TR}\right) = \left(1 + \mathbb{E}_t\left(R^*(r^*(s), s, s+\Delta)^{TR}\right)\right)\frac{\mathcal{E}^e_{s+\Delta}}{\mathcal{E}^e_s}$$

to formulate a second type of expectation for the devaluation rate

$$\begin{aligned}
\mathbb{E}^2\left(d_s\right) &= \left(\frac{\mathcal{E}^e_{s+\Delta} - \mathcal{E}^e_s}{\mathcal{E}^e_s}\right)_2 && (19)\\
&= \frac{\frac{1+\mathbb{E}_t(R(r(s),s,s+\Delta)^{TR})}{1+\mathbb{E}_t(R^*(r^*(s),s,s+\Delta)^{TR})}\mathcal{E}^e_s - \mathcal{E}^e_s}{\mathcal{E}^e_s} && (20)\\
&= \frac{\mathbb{E}_t(R(r(s), s, s+\Delta)^{TR}) - \mathbb{E}_t(R^*(r^*(s), s, s+\Delta)^{TR})}{1 + \mathbb{E}_t(R^*(r^*(s), s, s+\Delta)^{TR})}, && (21)
\end{aligned}$$

where the index $(\cdot)_2$ is introduced for later purposes. Obviously, the difference of the expected total one period returns determines the expected depreciation rate for any time s. According to the simple, risk-neutral efficient markets hypothesis the relevant interest rate differential is the optimum predictor of the future exchange rate depreciation.

Moreover, the depreciation rates calculated above can be used to calculate the expected exchange rate. Using the spot exchange rate at the current time t the expectations can be calculated iteratively as follows:

$$\mathcal{E}^e_{s+\Delta} = \frac{\mathbb{E}_t(R(r(s), s, s+\Delta)^{TR}) - \mathbb{E}_t(R^*(r^*(s), s, s+\Delta)^{TR})}{1 + \mathbb{E}_t(R^*(r^*(s), s, s+\Delta)^{TR})}\mathcal{E}^e_s + \mathcal{E}^e_s. \qquad (22)$$

3.3 Comparison of the two approaches

The two different expected exchange rates and expected depreciation rates, respectively, merit closer examination. This poses the question of whether these two expectations are different, show a similar structure or are even the same. While we use the UIP in the first approach directly, in the second approach the UIP in expectation is applied to short term investments in the future, which are stringed together to form the entire expected exchange rate term structure.

The expectations hypothesis leads to the conclusion that both expected depreciation rates (and consequently also the expected exchange rates) need to be the same. If one considers the opportunity of investing in either bonds in country X or in bonds in country Y, the expectations hypothesis states that the return from holding a long-term bond, e.g. with time-to-maturity $s + \Delta - t$, is the same as rolling over a sequence of short-term bonds (here short-term is equivalent to one period). Consequently, the expected exchange rate at time $s + \Delta - t$ is the same if one holds a long-term bond with time-to-maturity $s + \Delta - t$ or reinvests in short-term bonds repeatedly. The same argument is assumed to be valid for a long-term bond with time-to-maturity $s - t$. Hence, the calculated depreciation rates using the term structure needs to be the same as the depreciation rate resulting from the difference of the expected short term returns at time s, such that:

$$\left(\frac{\mathcal{E}^e_{s+\Delta} - \mathcal{E}^e_s}{\mathcal{E}^e_s}\right)_1 = \left(\frac{\mathcal{E}^e_{s+\Delta} - \mathcal{E}^e_s}{\mathcal{E}^e_s}\right)_2 . \tag{23}$$

Considering the liquidity preference, however, the assumed equality (23) of the expected depreciation rates given the expected exchange rate stemming from the term structure and the expected depreciation rates based on the expectation regarding the one period total return differentials may be not valid. This follows from the fact that the equality of yields assumed by the expectations hypothesis is not valid if a liquidity preference exists.

In section 3.4, we investigate various examples of the term structure of both expected depreciation rates and both expected exchange rates. It will become obvious that neither an absolutely similar nor an absolutely different structure can be observed, independently from the choice of the factors. Moreover, in some examples a similar development can be observed, whereas in other examples the paths differ from each other significantly.

3.4 Examples: Analysis and Visualization

In this section, the results acquired in section 3.1 and section 3.2 are used to interpret the term structure, the expected exchange rate, and the two elaborated expected depreciation rates of various examples of two economies with particular values for the several variables. Furthermore, the reader is referred to Cox et al. (1985b) for the characteristics of the term structure dependent on the choice and change of the particular parameters. Each choice of the values of the variables takes into consideration the various conditions stemming from the model of the economy. Furthermore, the examples are chosen such that the long-term yields are the same at the most part to simplify the sensitivity analysis. The term structure, the expected exchange rates and the expected depreciation rates are visualized.[3]

Example 1:

Variable	Country X	Country Y
r:	0.055 (5.5%)	0.055 (5.5%)
κ:	0.8	0.5
θ:	0.03 (3%)	0.03 (3%)
σ:	0.5	0.5
λ:	-0.4	-0.4
$\frac{2\kappa\theta}{\gamma+\kappa+\lambda}$:	0.0396 (3.96%)	0.0368 (3.68%)
$\frac{\kappa\theta}{\kappa+\lambda}$:	0.06 (6%)	0.15 (15%)

Table 1: Example 1

The economies of both countries are very similar. They only differ from

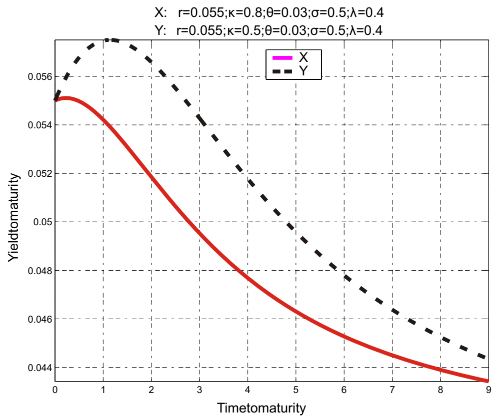
[3]For deeper understanding of the structure of the expected future one period total returns refer to Section 7.3.

each other in the difference of the parameters describing the respective speed of adjustment of the spot rate. Figure 1(a) shows the term structure in both countries.

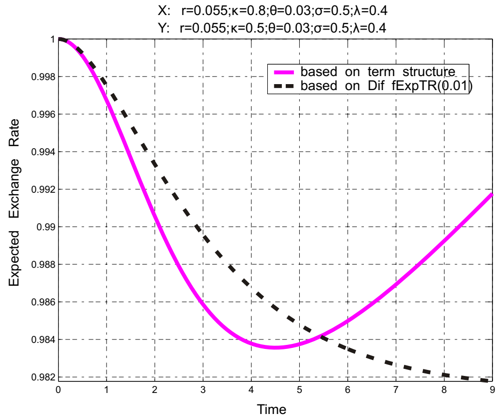
Approach 1: (Based on term structure) As one can see, although the spot rates are the same, the difference in κ leads to different term structures and different values of the long-term yields (11). Apparently, the spot rate of country X is humped and in excess of the long-term yield. The term structure of country Y is also humped, however, the maximum is lower than the maximum of country X's term structure. This difference accounts for the appreciation of the expected exchange rate (purple line) as shown in Figure 1(b). The appreciation reaches its maximum approximately at time 4.5, depending on the difference of the interest rates and the time to maturity. Consequently, the expected rate of depreciation becomes positive at this point of time (see figure 1(c)). In the long run, the expected exchange rate will return to unity due to the small difference in the long term yields.

Approach 2: (Based on expected spot rates) The green line describes the development of the expected rate of depreciation based on the difference of the expected spot rates. The spot rate of country Y is expected to return faster to its long term value θ and is thus expected to remain lower than country X's spot rate. Consequently, the green line in Figure 1(b) shows a constant appreciation. It is not surprising that the rate seems to become zero in the long-run as the expectation in the long-run can be approximated by $e^{\theta \cdot 0.01} - 1$ in both countries. Over all the expected depreciation rate based on the term structure seems to be more volatile.

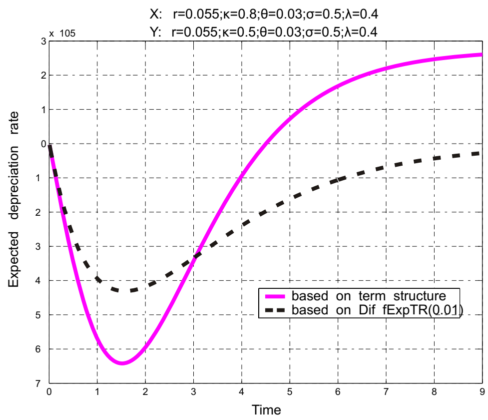
The phenomenon of an appreciation in the beginning, followed by a depreciation is not unknown at all. Overshooting and undershooting of the exchange rate is observed on foreign exchange markets as well as it is common to standard economic models. One explanation is that the equilibrium on the foreign exchange rate market is reached faster than one on the market for goods (see e.g. Krugman and Obstfeld (2003)).



(a) Term Structure



(b) Expected Exchange Rate



(c) Expected Depreciation Rate

Figure 1: Example 1

Example 2:

The economies in this example share identical spot rates and identical long-term yields (11). However, the long-term expected value of the spot rate is higher in country Y and the market value of risk is lower.

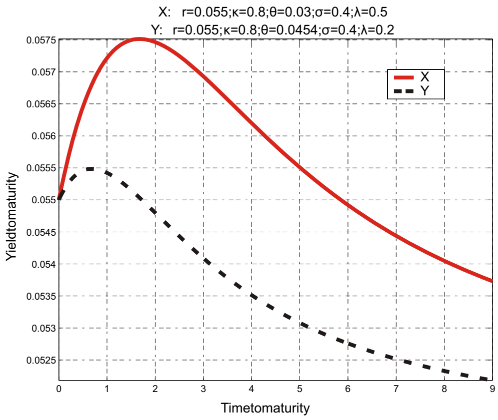
Approach 1: (Based on term structure) Figure 2(a) shows the term structure in both countries. The lower value of $|\lambda^*|$ of country Y leads to lower yields, although the expected spot rates are higher. The absolute effect of the difference in λ seems to be stronger than the effect of the difference in θ, as the higher value of θ^* would indicate higher yields in country Y compared to the yields paid in country X. The higher yields in country X are the reason why the exchange rate is expected to depreciate (see figure 2(b)). As a result, the expected rate of

Variable	Country X	Country Y
r:	0.055 (5.5%)	0.055 (5.5%)
κ:	0.8	0.8
θ:	0.03 (3%)	0.0454 (4.54%)
σ:	0.4	0.4
λ:	-0.5	-0.2
$\frac{2\kappa\theta}{\gamma+\kappa+\lambda}$:	0.0510 (5.1%)	0.0510 (5.1%)
$\frac{\kappa\theta}{\kappa+\lambda}$:	0.08 (8%)	0.0605 (6.05%)

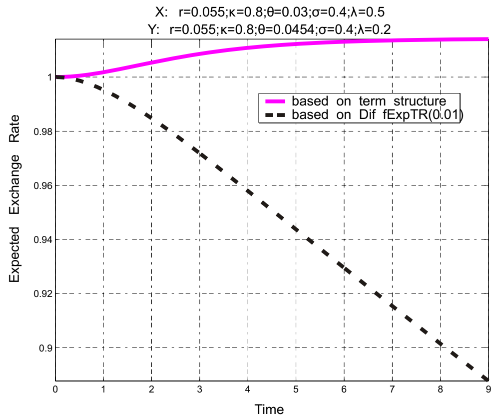
Table 2: Example 2

depreciation is positive as the purple line in Figure 2(c) verifies. In the long-run (not visible in figure2) yields converge. Thus the long-run expected exchange rate returns to the the spot exchange rate at time $t = 0$. The fact that κ and κ^*, the respective speed of adjustment, and the long-term yields are the same in both economies explains why the expected rate of depreciation becomes almost zero.

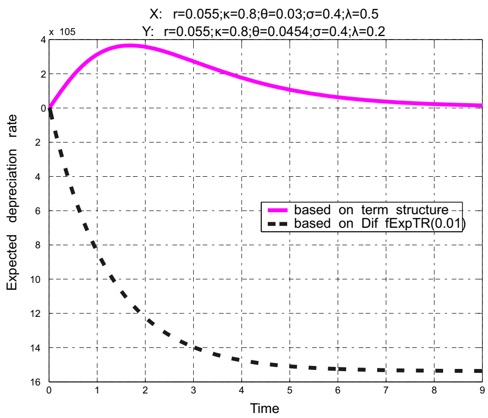
Approach 2: (Based on expected spot rates) The expected spot rate is higher in country Y for $t > 0$. Thus the expected rate of depreciation based on the expected future one period total returns is strictly negative (see green line in figure 2(c)). It converges to approximately $e^{(\theta-\theta^*)\cdot 0.01} - 1$. As a consequence, the expected exchange rate is appreciating (see green line in figure 2(b)). It is completely at odds with the first expectation.



(a) Term Structure



(b) Expected Exchange Rate



(c) Expected Depreciation Rate

Figure 2: Example 2

Example 3:

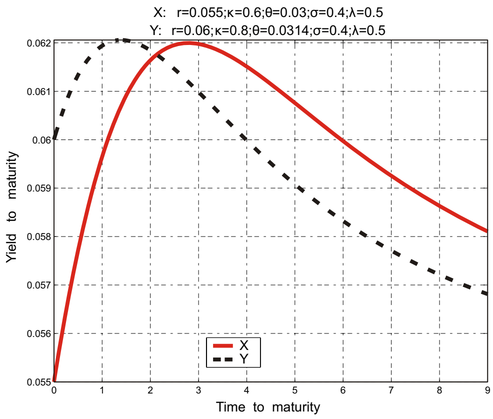
Variable	Country X	Country Y
r:	0.055 (5.5%)	0.06 (6%)
κ:	0.6	0.8
θ:	0.03 (3%)	0.0314 (3.14%)
σ:	0.4	0.4
λ:	-0.5	-0.5
$\frac{2\kappa\theta}{\gamma+\kappa+\lambda}$:	0.0534 (5.34%)	0.0534 (5.34%)
$\frac{\kappa\theta}{\kappa+\lambda}$:	0.18 (18%)	0.0837 (8.37%)

Table 3: Example 3

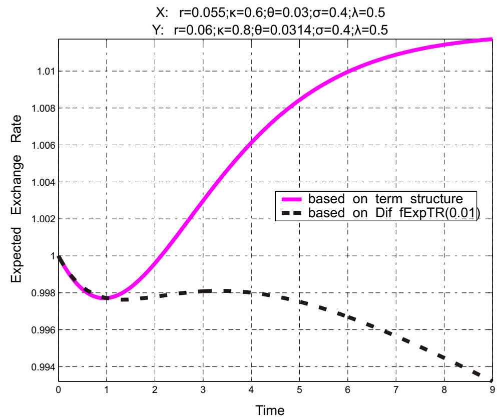
The economies in the last example differ with respect to the short horizon monetary parameters in our model. The current spot rate, its long term mean, and the speed of adjustment of the current rate to its average in country X are below the values in country Y. However, the long-term yields (11) are the same in both countries.

Approach 1: (Based on term structure) Figure 3 (a) shows the humped term structure in both countries. In the long-run (not visible in figure 2) the convergence of the yields in both countries leads to a convergence of the long-run expected exchange rate to the spot exchange rate at time $t = 0$. For short horizons however, the lower yields in country Y lead to an expected appreciation. While for medium horizons, the slower adjustment of the spot interest rate leads to higher yields in country Y and subsequently to the expectation of an depreciation in the medium horizon. Figure 3 (b) shows this expectations. As a result, the expected rate of depreciation which is negative in the short-run, becomes positive, and dies out in the long-run as the purple line in Figure 3 (c) verifies.

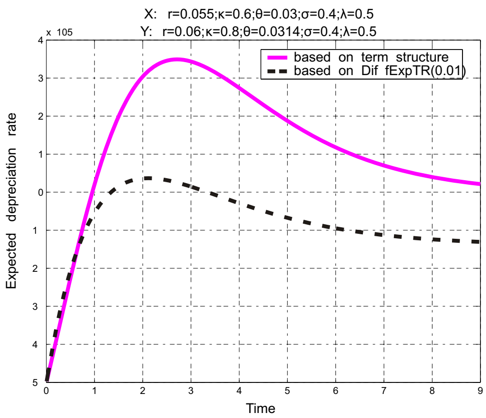
Approach 2: (Based on expected spot rates) 3 (c) shows that the expected rates of depreciation based on the expected future one period total returns show a similar development. However, due to the difference in the long-run average of the spot rates, the expected depreciation rate converges to a level significantly below zero, i.e. the exchange rate is expected to appreciate continuously in the long run (compare the green line in 3 (b)).



(a) Term Structure



(b) Expected Exchange Rate



(c) Expected Depreciation Rate

Figure 3: Example 3

4 The Influence of the Expectations on the Spot Exchange Rate

There are several approaches describing the reactions of both the spot exchange rate and the future expectations to new information. One possible way to argue is to assume that because of the expected development of the exchange rate, e.g. a depreciation of the currency, the agents anticipate the development by speculative purchases of the foreign currency. Hence, the spot exchange rate is influenced by the expectations. On the contrary, one could also argue that because of risk-averse agents the expected development is not anticipated by the agents.

4.1 The idea

This section is dedicated to the question of how changes of the factors of the economies influence the expectations and thus the spot exchange rate. However, we confine ourselves to the most prominent case in international economics: the reaction function of the spot rate to interest rate changes. The other variables are assumed to remain unchanged.

It is well-known that changes of the expected exchange rate lead to changes of the spot exchange rate. Most known models, however, only take one expected exchange rate into consideration when calculating the effect on the spot exchange rate. In our approach the whole path will influence the spot exchange rate. As a standard assumption we regard expectations as more rigid than the spot exchange rate. This rigidity is implemented by determining the spot rate as the minimizer of the change in the term structure of the exchange rate expectation. Figure 4 displays the idea of choosing a particular spot exchange rate such that the differences between the old and the new expected exchange rates are minimized.

In order to ensure that these differences do not balance themselves out, the

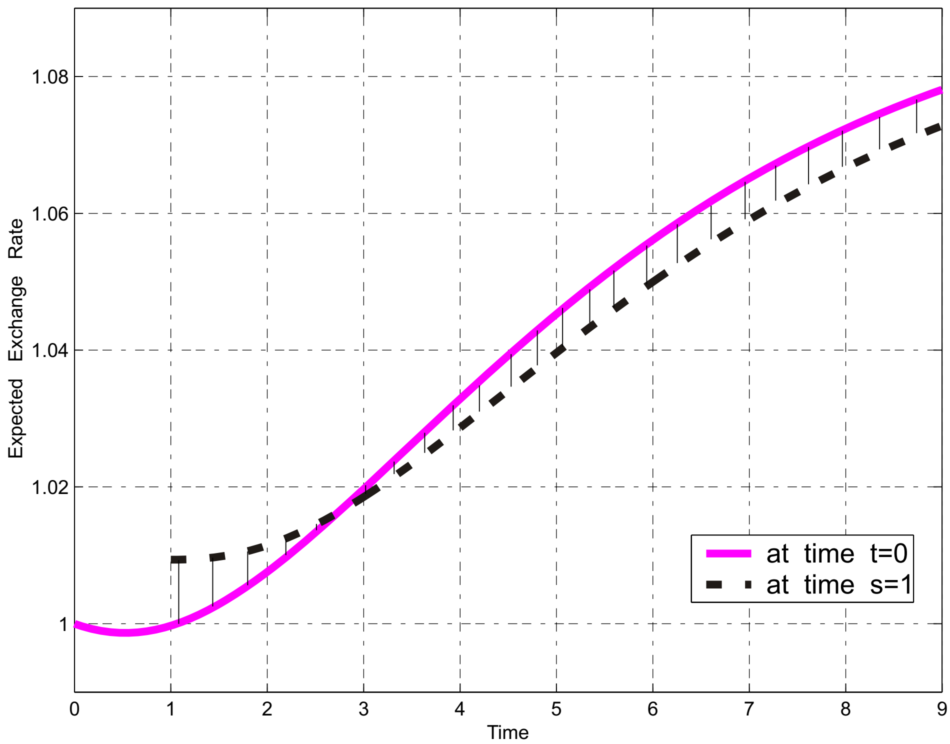


Figure 4: The spot exchange rate in the future

differences are squared. Furthermore, the expectations of the distant future are downweighted by introducing the weight $\frac{1}{T}$. As can be seen in the figure above, the spot rates were initially 9.5% and 10% respectively. At time $s = \Delta$ both countries loosen their monetary policy and both spot rates settle at 9%. The spot exchange rate adjusts, such that the marked area (in accordance with the weights) is minimized. As mentioned above, the other variables describing both economies are assumed to remain unchanged. Consequently, our approach analyzes the reaction of the spot exchange rate to sudden changes in the underlying variables.

In technical terms these assumptions can be expressed as follows:

$$\mathcal{E}_s = \arg\min \left\{ \int_s^\infty \left[\mathbb{E}_t(\mathcal{E}_T|\mathcal{E}_t) - \mathbb{E}_s(\mathcal{E}_T|\mathcal{E}_s)\right]^2 \frac{1}{T} dT \right\} \quad (24)$$

$$= \arg\min \left\{ \int_s^\infty \left[\frac{P^*(r^*(t), t, T)}{P(r(t), t, T)} \mathcal{E}_t - \frac{P^*(r^*(s), s, T)}{P(r(s), s, T)} \mathcal{E}_s \right]^2 \frac{1}{T} dT \right\}, \quad (25)$$

where $\mathbb{E}_s(\mathcal{E}_T|\mathcal{E}_s)$ stands for the expected exchange rate at time T given the information at time s, that is the current interest rate and the spot exchange rate $\mathcal{E}_s$.

The definite integral

$$F(y) = \int_a^b f(x,y)dx \tag{26}$$

is called *integral with parameter*. If the function is defined within an interval $[c,e]$ and the integrand is continuous within $[a,b] \times [c,e]$ and is partially differentiable with respect to y, then

$$\frac{d}{dy}\int_a^b f(x,y)dx = \int_a^b \frac{\partial f(x,y)}{\partial y}dx \tag{27}$$

for an arbitrary y within the interval $[c,e]$. A minimum of (26) needs to satisfy the following conditions:

$$F'(y) = \int_a^b \frac{\partial f(x,y)}{\partial y}dx = 0 \tag{28}$$

$$F''(y) = \int_a^b \frac{\partial^2 f(x,y)}{\partial y^2}dx > 0. \tag{29}$$

The initial type of problem can be interpreted as finding the argument minimizing the integral on the right hand side of (25), which depends on the parameter $\mathcal{E}_s$.[4] If one considers that the integrand of (25) is a continuous function within the investigated interval and can be partially differentiated with respect to $\mathcal{E}_s$, the sufficient conditions (28) and (29) can be used. The conditions can be written

[4]Although the integral in (25) is an improper integral as the upper limit is infinite, we restricted our investigation of the solution of that integral to a closed interval $[t,R]$ for technical reasons. We hold the view that this restriction does not alter the usefulness of that approach, as the value of R may be arbitrarily large and the impact of not included expectations disappears as the horizon tends to infinity. Moreover, the weight decreases the importance of those expectations for the determination of $\mathcal{E}_s$. Finally, the restriction of the investigation to a close interval is reasonable for computational purposes.

as follows:

$$\int_s^R \frac{-2}{T}\left[\frac{P^*(r^*(t),t,T)}{P(r(t),t,T)}\mathcal{E}_t - \frac{P^*(r^*(s),s,T)}{P(r(s),s,T)}\mathcal{E}_s\right]\frac{P^*(r^*(s),s,T)}{P(r(s),s,T)}dT = 0 \quad (30)$$

$$\int_s^R \frac{2}{T}\frac{P^*(r^*(s),s,T)^2}{P(r(s),s,T)^2}dT > 0. \quad (31)$$

It can be easily seen that (31) is satisfied. Consequently, the spot exchange rate at time s can be calculated by:

$$\mathcal{E}_s = \frac{\int_s^R \frac{P^*(r^*(t),t,T)}{P(r(t),t,T)}\frac{P^*(r^*(s),s,T)}{P(r(s),s,T)}\frac{\mathcal{E}_t}{T}dT}{\int_s^R \frac{P^*(r^*(s),s,T)^2}{P(r(s),s,T)^2}\frac{1}{T}dT}. \quad (32)$$

Finding an analytical solution of the integrals presented in (32) is not a trivial problem. Using the strategy of adaptive quadrature, however, the value can easily be calculated numerically.

The solution $\mathcal{E}_s$ for the spot rate can also be interpreted as the expectation of the spot rate at time s given the information at time t and conditional on the change of the underlying economic variable used to calculate $\mathcal{E}_s$. If instead of an actual spot rate at time s the expectation of the spot rate at time s given the spot rate at the current time t (equation (40)) is used, the value of (32) can be interpreted as another expectation of the exchange rate at time s. It follows:

$$\mathcal{E}_s^e = \frac{\int_s^R \frac{P^*(r^*(t),t,T)}{P(r(t),t,T)}\frac{P^*(E(r^*(s)|r^*(t)),s,T)}{P(E(r(s)|r(t)),s,T)}\frac{\mathcal{E}_t}{T}dT}{\int_s^R \frac{P^*(E(r^*(s)|r^*(t)),s,T)^2}{P(E(r(s)|r(t)),s,T)^2}\frac{1}{T}dT}. \quad (33)$$

Investigations of these expectations were not done. However, the reader may take into consideration that this approach would be another alternative to the expectations formed in section 3.1 and 3.2.

4.2 New Expectations and the Adjustment of the Spot Exchange Rate

In this section, we present two examples of how observed values of the spot rate at time $s = \Delta$ influence the spot exchange rate and the expectations with regard to the results of section 4.1. First, we chose the economies presented in Example 3. Initially, the spot rates at time $t = 0$ were 5.5% in both countries. Figure 5 displays the results when at time $s = 1$ one can observe the spot rate to be risen to 6% in country Y, while it has not changed in country X. As the other variables describing both economies have not changed, the long-term yields (11) of both economies have not changed either.

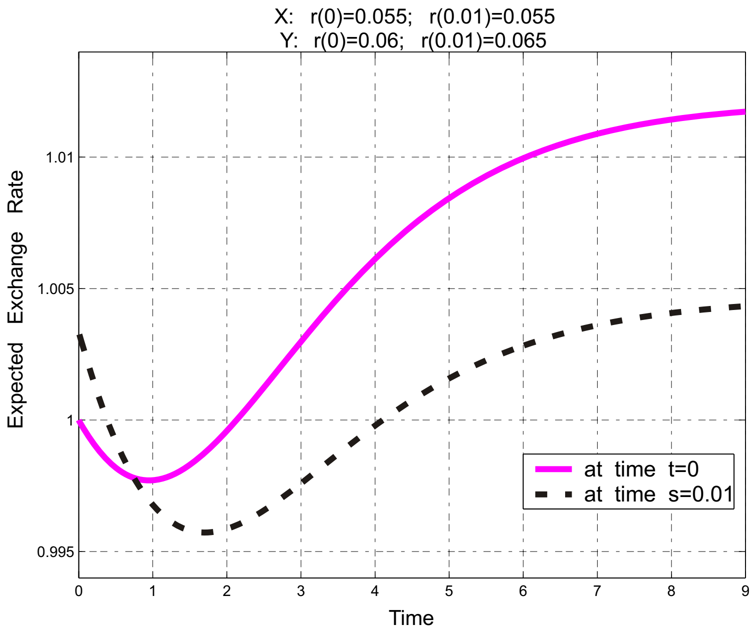


Figure 5: Example 7

The structure of the expectations, however, differs from the initial structure, because different values of the spot rate influence the prices of the bond. Equation (7) shows that changes of r result in changes of the price. Furthermore, the different values of the spot exchange rate at time $s = \Delta$ also influence the expectations (see equation (16)). One would expect an exchange rate that is above its initial level, as the spot rate in country Y increased to a higher level. Consequently, one would expect a higher demand for the currency of country Y which results in a depreciation of the currency of country X and thus an increase in the exchange rate. Obviously, the calculated spot exchange rate using (32) does meet this expectation. Additionally, it ensures that the squared weighted differences of the expectations formed at time $t = 0$ and those formed at time $s = \Delta$ are minimized. To ensure that the expectations almost stay the same, the spot exchange rate adjusts to $E_1 = 1.0033$. This example displays a situation where the change of interest rates and the resulting implication on the spot exchange rate is not expected initially. At time $t = 0$ an appreciation is expected. The increase of country Y's interest rate leads to a depreciation.

Secondly, we chose the economies presented in Example 2. An increase in the spot rate in country Y from 5.5% to 6% leads to an appreciation of its currency. The spot exchange rate rises to $E_1 = 1.0026$. Figure 6 illustrates this example. In this situation the direction of the change of the spot exchange rate at time $s = \Delta$, that is either a depreciation or an appreciation, is consistent with the initial expectation.

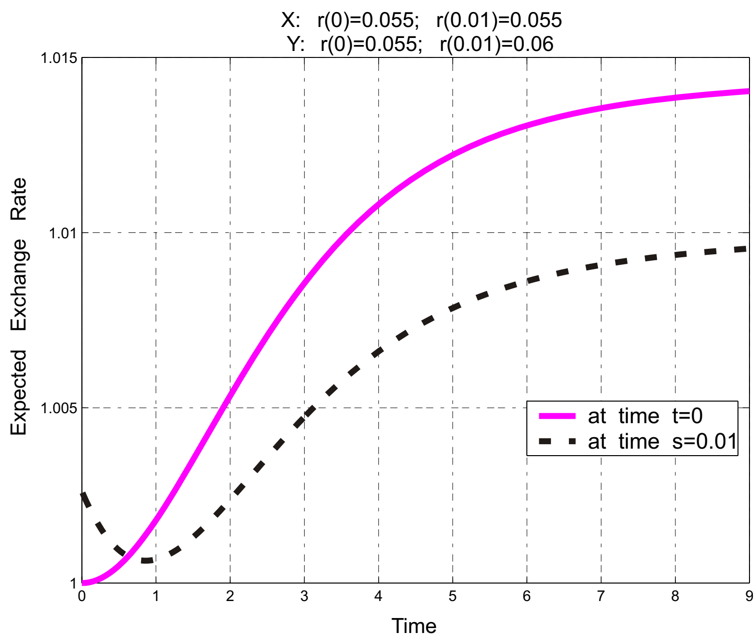


Figure 6: Example 8

5 Calculation of the ETR (Δ)

In order to investigate the characteristics of the expected total one period return (ETR (Δ)) mentioned above, the expectation needs to be given explicitly in a form dependent on the variables which may influence the behavior. Basically, an explicit form of

$$\begin{aligned} \mathbb{E}_t(R(r(s), s, s+\Delta)^{TR}) &= \mathbb{E}_t(e^{R(r(s),s,s+\Delta)\Delta} - 1) \\ &= \mathbb{E}_t(e^{r(s)B(s,s+\Delta)-ln(A(s,s+\Delta))}) - 1 \end{aligned} \tag{34}$$

for a particular future time s is needed.[5]

$A(s, s+\Delta)$ and $B(s, s+\Delta)$ are defined as in (8) and (9) respectively. Obviously $A(s, s+\Delta)$ and $B(s, s+\Delta)$ are constant for any choice of s and independent from the current time t and the spot rate $r(t)$. For an easier notation, $A(s, s+\Delta)$ and $B(s, s+\Delta)$ are replaced by $\bar{A}$ and $\bar{B}$ respectively.

Cox et al. (1985b) show that the probability density of the spot interest rate r at time s, conditional on its value at the current time t, is given by:

$$f(r(s), s; r(t), t) = ce^{-u-v}\left(\frac{v}{u}\right)^{\frac{q}{2}} I_q(2(uv)^{\frac{1}{2}}), \tag{35}$$

where

$$c \equiv \frac{2\kappa}{\sigma^2(1-e^{-\kappa(s-t)})} \tag{36}$$

$$u \equiv cr(t)e^{-\kappa(s-t)} \tag{37}$$

$$v \equiv cr(s) \tag{38}$$

$$q \equiv \frac{2\kappa\theta}{\sigma^2} - 1 \tag{39}$$

and $I_q(\cdot)$ is the *modified Bessel function of the first kind of order* q. The distribution function is the *noncentral chi-square*, $\chi^2[2cr(s); 2q+2, 2u]$, with $2q+2$ *degrees of freedom* and *parameter of noncentrality* $2u$ proportional to the current

[5] Note that for $s = t$ the expectation is already known. It is equivalent to the total return of a bond with time-to-maturity $T - t = \Delta$, which is known from the term structure. When dealing with the expectation, therefore, one can assume $s > t$.

spot rate. Straightforward calculations give the expected value and variance of $r(s)$ as:

$$\begin{aligned}
\mathbb{E}(r(s) \mid r(t)) &= r(t)e^{-\kappa(s-t)} + \theta(1 - e^{-\kappa(s-t)}) && (40)\\
var(r(s) \mid r(t)) &= r(t)\left(\frac{\sigma^2}{\kappa}\right)(e^{-\kappa(s-t)} - e^{-2\kappa(s-t)}) && (41)\\
& \quad +\theta\left(\frac{\sigma^2}{2\kappa}\right)(1 - e^{-\kappa(s-t)})^2. && (42)
\end{aligned}$$

The current interest rate $r(t)$ influences the value of u, which is defined as in (37).

First of all, the probability density of a distribution function, which is non-central chi-square with n degrees of freedom and a parameter of noncentrality of $\lambda > 0$, needs to satisfy the following condition:

$$\int_0^\infty \frac{1}{2}e^{-\frac{x+\lambda}{2}}\left(\frac{x}{\lambda}\right)^{\frac{n-2}{4}} I_{\frac{n-2}{2}}((\lambda x)^{\frac{1}{2}})dx = 1. \tag{43}$$

With

$$\begin{aligned}
\bar{v} &= v - r(s)\bar{B}\\
&= r(s)(c - \bar{B}) && (44)\\
\bar{u} &= \frac{uv}{\bar{v}}\\
&= \frac{uc}{c - \bar{B}}, && (45)
\end{aligned}$$

where $\bar{u} > 0$, and (35), the expectation of $e^{r(s)\bar{B}-ln(\bar{A})}$ can be written as follows:

$$\begin{aligned}
& \mathbb{E}_t(e^{r(s)\bar{B}-ln(\bar{A})})\\
= & \int_0^\infty e^{r(s)\bar{B}-ln(\bar{A})}ce^{-u-v}\left(\frac{v}{u}\right)^{\frac{q}{2}} I_q(2(uv)^{\frac{1}{2}})dr(s) && (46)\\
= & \int_0^\infty \frac{c}{\bar{A}}e^{r(s)\bar{B}-u-v\left(\frac{v}{u}\right)^{\frac{q}{2}}} I_q(2(\bar{u}\bar{v})^{\frac{1}{2}})dr(s)\\
= & \int_0^\infty \frac{c}{\bar{A}}e^{\bar{u}-u}e^{-\bar{u}-\bar{v}\left(\frac{v}{u}\right)^{\frac{q}{2}}} I_q(2(\bar{u}\bar{v})^{\frac{1}{2}})dr(s)\\
= & \int_0^\infty \frac{c}{\bar{A}}\left(\frac{1}{c}\right)^{-q}\left(\frac{1}{c}\right)^{q}(c-\bar{B})^{-q}(c-\bar{B})^{q}e^{\bar{u}-u}e^{-\bar{u}-\bar{v}}\left(\frac{v}{u}\right)^{\frac{q}{2}} I_q(2(\bar{u}\bar{v})^{\frac{1}{2}})dr(s)\\
= & \int_0^\infty \frac{c}{\bar{A}}\left(\frac{1}{c}\right)^{-q}(c-\bar{B})^{-q}e^{\bar{u}-u}e^{-\bar{u}-\bar{v}}\left(\frac{r(s)(c-\bar{B})}{\frac{uc}{c-\bar{B}}}\right)^{\frac{q}{2}} I_q(2(\bar{u}\bar{v})^{\frac{1}{2}})dr(s)
\end{aligned}$$

$$= \int_0^\infty \frac{c}{\bar{A}} \left(\frac{c - \bar{B}}{c} \right)^{-q} e^{\bar{u}-u} e^{-\bar{u}-\bar{v}} \left(\frac{\bar{v}}{\bar{u}} \right)^{\frac{q}{2}} I_q(2(\bar{u}\bar{v})^{\frac{1}{2}}) dr(s). \tag{47}$$

Using the substitution $n = 2q + 2$, $\lambda = 2\bar{u}$, and $x = 2\bar{v}$ and considering the chain rule of differentiation, (43) is equivalent to

$$\begin{aligned}
1 &= \int_0^\infty \frac{1}{2} e^{-\frac{x+\lambda}{2}} \left(\frac{x}{\lambda} \right)^{\frac{n-2}{4}} I_{\frac{n-2}{2}}((\lambda x)^{\frac{1}{2}}) dx \\
&= \int_0^\infty \frac{1}{2} e^{-\frac{x+\lambda}{2}} \left(\frac{x}{\lambda} \right)^{\frac{q}{2}} I_q((\lambda x)^{\frac{1}{2}}) dx \qquad (48) \\
&= \int_0^\infty \frac{1}{2} e^{-\frac{x+2\bar{u}}{2}} \left(\frac{x}{2\bar{u}} \right)^{\frac{q}{2}} I_q((2\bar{u}x)^{\frac{1}{2}}) dx \\
&= \int_0^\infty \frac{1}{2} e^{-\bar{u}-\bar{v}} \left(\frac{\bar{v}}{\bar{u}} \right)^{\frac{q}{2}} I_q(2(\bar{u}\bar{v})^{\frac{1}{2}}) 2dv \\
&= \int_0^\infty e^{-\bar{u}-\bar{v}} \left(\frac{\bar{v}}{\bar{u}} \right)^{\frac{q}{2}} I_q(2(\bar{u}\bar{v})^{\frac{1}{2}})(c - \bar{B}) dr(s). \qquad (49)
\end{aligned}$$

As a result, considering (49) together with the fact that $A(s, s+\Delta)$, $B(s, s+\Delta)$, u, $\bar{u}$, and c are constant, the expectation (46) can be written as:

$$\begin{aligned}
&\mathbb{E}_t(e^{r(s)\bar{B}-ln(\bar{A})}) \\
&= \int_0^\infty \frac{c}{\bar{A}} \left(\frac{c - \bar{B}}{c} \right)^{-q} e^{\bar{u}-u} e^{-\bar{u}-\bar{v}} \left(\frac{\bar{v}}{\bar{u}} \right)^{\frac{q}{2}} I_q(2(\bar{u}\bar{v})^{\frac{1}{2}}) dr(s) \qquad (50) \\
&= \frac{c}{\bar{A}} \left(\frac{c - \bar{B}}{c} \right)^{-q} e^{\bar{u}-u} \frac{1}{c - \bar{B}} \int_0^\infty e^{-\bar{u}-\bar{v}} \left(\frac{\bar{v}}{\bar{u}} \right)^{\frac{q}{2}} I_q(2(\bar{u}\bar{v})^{\frac{1}{2}})(c - \bar{B}) dr(s) \\
&= \frac{c}{\bar{A}} \left(\frac{c - \bar{B}}{c} \right)^{-q} e^{\bar{u}-u} \frac{1}{c - \bar{B}} \\
&= \frac{1}{\bar{A}} \left(\frac{c}{c - \bar{B}} \right)^{q+1} e^{\frac{u\bar{B}}{c-\bar{B}}} \\
&= \frac{1}{A(s, s+\Delta)} \left(\frac{c}{c - B(s, s+\Delta)} \right)^{q+1} e^{\frac{uB(s,s+\Delta)}{c-B(s,s+\Delta)}}. \qquad (51)
\end{aligned}$$

Hence, the expression (34) is equivalent to:

$$\mathbb{E}_t(R(r(s), s, s+\Delta)^{TR}) = \frac{e^{\frac{uB(s,s+\Delta)}{c-B(s,s+\Delta)}}}{A(s, s+\Delta)} \left(\frac{c}{c - B(s, s+\Delta)} \right)^{q+1} - 1.$$

Furthermore, expression (19) can be written as:

$$\left(\frac{\mathcal{E}^e_{s+\Delta} - \mathcal{E}^e_s}{\mathcal{E}^e_s} \right)_2 = \frac{\frac{1}{\bar{A}} \left(\frac{c}{c-\bar{B}} \right)^{q+1} e^{\frac{u\bar{B}}{c-\bar{B}}} - \frac{1}{\bar{A}^*} \left(\frac{c^*}{c^*-\bar{B}^*} \right)^{q^*+1} e^{\frac{u^*\bar{B}^*}{c^*-\bar{B}^*}}}{\frac{1}{\bar{A}^*} \left(\frac{c^*}{c^*-\bar{B}^*} \right)^{q^*+1} e^{\frac{u^*\bar{B}^*}{c^*-\bar{B}^*}}}$$

$$= \frac{\bar{A}^*}{\bar{A}} \left(\frac{c}{c - \bar{B}} \right)^{q+1} \left(\frac{c^* - \bar{B}^*}{c^*} \right)^{q^*+1} e^{\frac{u\bar{B}}{c-\bar{B}} - \frac{u^*\bar{B}^*}{c^*-\bar{B}^*}} - 1.$$

Calculations lead to an expected future one period total return in the long-run of:

$$\begin{aligned} Yl &= \lim_{s\to\infty} \frac{1}{A(s,s+\Delta)} \left(\frac{c}{c - B(s,s+\Delta)} \right)^{q+1} e^{\frac{uB(s,s+\Delta)}{c-B(s,s+\Delta)}} - 1 \\ &= \frac{1}{\bar{A}} \left(\frac{\frac{2\kappa}{\sigma^2}}{\frac{2\kappa}{\sigma^2} - \bar{B}} \right)^{\frac{2\kappa\theta}{\sigma^2}} - 1. \end{aligned}$$

6 Summary

In this paper, the structure of the expected exchange rates was investigated. The emphasis was put on the influence of the fundamental factors of the economies on the expectations.

Based on the term structures resulting from the model of Cox et al. (1985b) we constructed a path of expected exchange rates under certain assumptions. Various investment opportunities of the agents based on the two different term structures led to an expectation of the future development of the spot exchange rate. Additionally, a further expectation was formed which was based on the assumptions that there are individuals who expect a certain exchange rate at a future time resulting from a possible repeated investment in short-term bonds. An explicit solution of that expectation dependent on the fundamental factors could be given. The behavior of this expectation dependent on changes in the variables, except for the variable describing the volatility of the spot rate, seems to be consistent with the behavior of the term structure stated in Cox et al. (1985b). Consequently, it was easy to find economic reasonable interpretations for the observed behavior. The lack of a possible explanation for the influence of the variance of the current interest rate and for the significance of the length of the period on the expected future one period total return may indicate deeper and further connections.

With the help of these expectations we calculated another exchange rate expectation with regard to future times. Both investigated expected depreciation rates partly show similar, almost identical behavior, however, partly absolutely different behavior. Nevertheless, all examples presented show reasonable structures of the expectations. This poses the question of which path of the expected exchange rate is more meaningful and reasonable respectively. We hold the view that the question cannot be answered easily as each argumentation underlying the particular expectation is reasonable. Consequently, it is hard to be in favor of one of the paths. It can rather be assumed that in the long-run both expectations are not meaningful at all as the influence of the different price levels, the inflation rate and the trading of goods are neglected. Within the framework of the purchasing power parity the influence of price levels gains more importance in the long-run. As a result, the presented approach may not provide reasonable statements with regard to the expected exchange rates in the long-run.

In addition to that, we presented a possibility of determining the spot exchange rate taking the whole path of expected exchange rates into consideration. This represents an extension to the more simple models explaining the influence of the expectation on the current value. Although an analytical solution could not be obtained, a way to calculate the spot exchange rate using numerical integration methods was shown.

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7 Appendix

7.1 Table of Symbols

The following table lists the most important symbols used throughout this work:

Symbol	Interpretation
t	current time; set to 0 (arbitrary)
s	future time; $s > t$
T	future time; date of maturity of zero bond
$T - t$	time-to-maturity of zero bond at current time t
κ	speed of adjustment of spot rate
θ	long-term value of spot rate
σ^2	interest rate variance
λ	market risk value
r, $r(t)$	current interest rate
$r(s)$	interest rate at future time s
$(\cdot)^*$	respective variable of country Y
$P(r, t, T)$	price of zero bond at time t with time-to-maturity $T - t$ and spot rate r
$R(r, t, T)$	yield of zero bond at time t with time-to-maturity $T - t$ and spot rate r
A, B	determinants of price and yield of a zero bond respectively
ϕ_1, ϕ_2, ϕ_3	determinants of price and yield of a zero bond respectively
$\mathcal{E}_0$	starting exchange rate; set to 1 (arbitrary)
$\mathcal{E}_t$	exchange rate at current time t
$\mathcal{E}_s$	exchange rate at future time s
$\mathcal{E}_s^e$	expected exchange rate at future time s
Δ	length of period; set to 0.01 in the latter part of this thesis (arbitrary)
d_s	rate of devaluation at time s
$\mathbb{E}_t$	expectation operator conditional on information an time t
$\mathbb{E}^i(d_s)$	expected rate of devaluation of approach $i = 1, 2$

7.2 The Liquidity Preference Hypothesis

The condition expressing the Liquidity Preference can be easily understood if the results of Cox et al. (1985b) are combined with the explicit form of the ETR (Δ) presented in Subsection 5

If one assumes that the expected future one period total return at time s can be approximated by $e^{E(r(s)|r(t))B(s,s-t)-ln(A(s,s-t))}-1$, where $E(r(s)|r(t))$ is calculated as in (40). As stated in section 2, the yield (10) approaches the spot rate as maturity nears. Consequently, the assumption of a liquidity preference for bonds with a maturity date in the sufficiently distant future can now be written as (see equation (13)):

$$e^{r(t)\Delta}e^{E(r(t+\Delta)|r(t))\Delta}\cdots e^{E(r(T)|r(t))\Delta} < e^{R(r,t,T)(T-t)}. \tag{52}$$

If the maturity date is in the sufficiently distant future, the yield-to-maturity $R(r,t,T)$ is approximately $R(r,t,\infty)$ and, considering (40), the expected spot rate given the current interest rate $E(r(T)|r(t))$ is approximately θ. If the inequation $R(r,t,\infty) > \theta$ is valid, there is a date of maturity T such that inequation (52) holds.

Although $e^{E(r(s)|r(t))B(s,s-t)-ln(A(s,s-t))}-1$ is not the correct expected future one period total return as

$$\mathbb{E}(e^{r(s)B(s,s-t)-ln(A(s,s-t))}) \neq e^{E(r(s)|r(t))B(s,s-t)-ln(A(s,s-t))},$$

it can be shown that for a sufficiently short length of one period the approximation

$$\mathbb{E}(e^{R(r(s),s,s+\Delta)\Delta}) \approx e^{E(r(s)|r(t))\Delta}, \tag{53}$$

can be proven valid. This can be seen if one considers a sufficiently small Δ such that

$$\begin{aligned} e^{\gamma\Delta} &\approx 1+\gamma\Delta \\ (\gamma+\kappa+\lambda)\Delta+2 &\approx 2 \end{aligned}$$

$$
\begin{aligned}
ln\left(\frac{c}{c-\Delta}\right) &\approx \frac{c}{c-\Delta}-1\\
\frac{\Delta}{c-\Delta} &\approx \frac{\Delta}{c}
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
B(s,s+\Delta) &= \frac{2(e^{\gamma\Delta}-1)}{(\gamma+\kappa+\lambda)(e^{\gamma\Delta}-1)+2\gamma}\\
&\approx \frac{2\Delta\gamma}{(\gamma+\kappa+\lambda)\Delta\gamma+2\gamma}\\
&= \frac{2\Delta}{(\gamma+\kappa+\lambda)\Delta+2}\\
&\approx \frac{2\Delta}{2}\\
&= \Delta
\end{aligned}
$$

and analogous to that

$$
A(s,s+\Delta) \approx 1.
$$

Hence,

$$
\mathbb{E}(R(r(s),s,s+\Delta)^{TR}) =
$$

$$
\begin{aligned}
&= \frac{1}{A(s,s+\Delta)}\left(\frac{c}{c-B(s,s+\Delta)}\right)^{q+1} e^{\frac{uB(s,s+\Delta)}{c-B(s,s+\Delta)}}-1\\
&\approx \left(\frac{c}{c-\Delta}\right)^{q+1} e^{\frac{u\Delta}{c-\Delta}}-1\\
&= e^{\frac{2\kappa\theta}{\sigma^2}ln(\frac{c}{c-\Delta})} e^{\frac{cr(t)e^{-\kappa(s-t)}\Delta}{c-\Delta}}-1\\
&\approx e^{\frac{2\kappa\theta}{\sigma^2}\left(\frac{c}{c-\Delta}-1\right)} e^{r(t)e^{-\kappa(s-t)}\Delta}-1\\
&= e^{\frac{2\kappa\theta}{\sigma^2}\frac{\Delta}{c-\Delta}} e^{r(t)e^{-\kappa(s-t)}\Delta}-1\\
&\approx e^{\frac{2\kappa\theta}{\sigma^2}\frac{\Delta\sigma^2(1-e^{-\kappa(s-t)})}{2\kappa}} e^{r(t)e^{-\kappa(s-t)}\Delta}-1\\
&= e^{r(t)e^{-\kappa(s-t)}+\theta(1-e^{-\kappa(s-t)})}-1\\
&= e^{E(r(s)|r(t))}-1.
\end{aligned}
$$

Consequently, the condition $R(r,t,\infty)>\theta$ also ensures a liquidity preference if the correct expected future one period total returns are investigated.

7.3 Characteristics of the ETR(Δ)

In this section, we present the results of the analysis of the ETR (Δ), $\Delta = 0.01$ based on the explicit form evolved in section 5. That is, the characteristics of

$$\mathbb{E}_t(R(r(s), s, s + \Delta)^{TR}) = \mathbb{E}_t(e^{r(s)B(s,s+\Delta)-ln(A(s,s+\Delta))}) - 1 \tag{54}$$

are investigated.

As mentioned above, the expected future one period total return at time $s = 0$ is already known from the term structure. Hence, the behavior of the expected yield at time $s = t = 0$ is also known.

Before presenting the results and visualizations of various examples, we theoretically show the validity of our results. First of all, we introduced several intervals for possible and economic reasonable values of the variables. As a starting point, we refer to Chatterjee (2004). In this paper, quasi-maximum likelihood estimates of the model parameters are obtained by using a Kalman filter to calculate the likelihood function. Furthermore, estimates of σ^2 presented by Brown and Dybwig (1986) were used to cut down the intervals to reasonable lengths. The bank base rates of the FED and the EZB of the last decades serve as a framework for the variables θ and r. Additionally, the conditions (1)-(6) on the variables from Cox et al. (1985b) were used. A further condition stemming from the assumption that there exists a liquidity preference can be written as:

$$\sigma < \sqrt{-2\kappa\lambda}$$

Table 4 gives an overview of the intervals.

We calculated the partial derivatives analytically, however, it was not trivial to prove an unambiguous behavior for any composition of the variables. Instead, we used the partial derivatives to give evidence considering the characteristics of (54) by calculating the maximum and minimum, respectively, for any composition of the variables within the intervals mentioned above and proved the continuity of the partial derivatives. A positive minimum indicates that the expectation increases with an increase in the particular variable, while a negative maximum

Variable	Interval	Increment
κ	[0.1,1]	0.05
θ	[0.005,0.08]	0.00025
r	[0.005,0.12]	0.00025
λ	[-(κ-0.05),-0.05]	0.05
σ	[0.05,min($\sqrt{-2\kappa\lambda}$,1)]	0.05

Table 4: Values of the variables of an economy

indicates that the expectation decreases with an increase in the particular variable. It is easy to show that all partial derivatives are continuous functions within the investigated intervals. This can be seen if one considers that any partial derivative of (54) is a combination of products, sums and fractions of the several elements of (54) and their partial derivatives. Considering the conditions on the variables, we split up equation (54) into several components and proved their continuity. According to the characteristics of continuous functions, the continuity of the several components and their partial derivatives, respectively, and the examination, whether the several parts and their various combinations are well defined, are sufficient for the proof of continuity. Consequently, we could easily show the continuity of function (54). These results substantiate our propositions.

At this point, we present the behavior of the expectation. Note that another choice of the length of one period, e.g. $\Delta = 0.1$, influences the value of the ETR (Δ) in the long-run. This follows from (54) for $s \to \infty$. The value increases as the term increases. Moreover, the expected future one period total return at any time increases as the term increases. The result is not surprising, as bonds with a longer maturities achieve a higher total return.

As mentioned in section 2, we assume that the inequation (13) holds and that the expected future one period total return can be approximated by $e^{E(r(s)|r(t))0.01} -$

1. Considering (40), for $s \to 0$ the ETR (Δ) can be approximated by $e^{r \cdot \Delta} - 1$ (for $s \to \infty$ it can be approximated by $e^{\theta \cdot \Delta} - 1$). Furthermore, for $\Delta = 0.01$ the approximation $e^{r \cdot 0.01} \approx 1 + r \cdot 0.01$ for small $r \cdot 0.01$ is used. According to Table 4, it can be assumed that the size of r is sufficiently small to allow for this approximation.

While the ETR (0.01) is rising if $r \cdot 0.01$ is below the value in the long-run, the ETR (0.01) is falling otherwise.

Moreover, several other comparative statics for the yield curve are obtained.

Calculations have shown that an increase in the current interest rate increases the ETR (0.01) at any future time s.[6] This can be easily interpreted if one considers that a bond's yield is a composition of the spot rate and a premium. A higher spot rate influences the expectations concerning the one period total returns, as a higher value of spot rate indicates greater yields. The long-term value of the spot rate, θ, has not changed and, therefore, the expected one period yields in the long-run have not changed very much. Hence, the effect is greater for the expectations in the relatively near future.

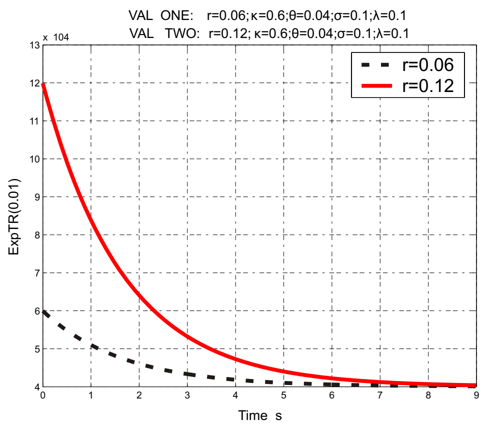

Figure 7: Effect of an increase in r on the ExpTR(0.01)

[6]The influence of the various parameters on the total return of a one period bond at time $s = 0$ is determined by the term structure. It coincides with the effects described here. An extensive analysis of this special case may be found in Cox et al. (1985b).

Similarly, an increase in the steady state mean θ increases the ETR (0.01). Here the effect is greater for the expectations in the relatively distant future as the long-term value θ has changed, whereas the current interest rate r has not.

The effect of a change in κ may be of either sign depending on the current interest rate, that is, the expected value is an increasing function of the speed of adjustment parameter κ if the spot rate is less than θ and a decreasing function of κ if the spot rate is greater than θ respectively. This can be seen if one considers that a higher value of κ means that the spot rate adjusts faster to a higher/lower level.

The ETR (0.01) decreases as λ increases. This can be easily seen as one remembers that higher values of λ indicate lower premiums as λ is the market value of risk. As λ increases (since λ is negative this corresponds to a decrease in $|\lambda|$), the value of risk decreases. This development influences the expectation of the agent. A lower market value of risk decreases the expected one period yield and, consequently, the total return. The influence on the total return of a one period bond at time $s = 0$ is the same.

Hence, for all named parameters the behavior of the expectation is consistent with the behavior of the term structure presented by Cox et al. (1985b).

The effect of an increase in σ^2, however, is not unambiguous: for the particular choice of the variables (see Figure 8) it increases the ETR (0.01). The effect of a change of σ^2 on the (expected) yield at time $s = 0$ is already known (see Cox et al. (1985b)). A higher value of the variance of the interest rate, σ^2, indicates more uncertainty about future real production opportunities, and thus more uncertainty about future consumption. As a consequence, the guaranteed claim in a bond is valued more highly by investors and the yield decreases. Compared with that, the ETR (0.01) increases at any time s.

Moreover, a change of the length of one period influences the impact of a

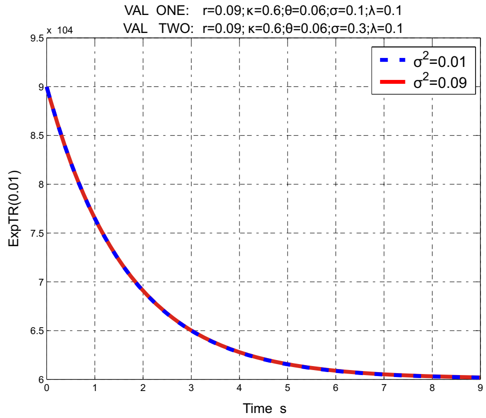

Figure 8: Effect of an increase in σ^2 on the ExpTR(0.01)

change in σ^2 on the ETR (0.01). Longer periods change the influence of changes in σ^2 on the expectations. For example, with $\Delta = 2.5$, and given the particular values of the variables as in Figure 8, an increase in σ^2 leads to a contrary statement. Here, an increase in the variance of the spot rate leads to a decrease of the ETR (2.5).

This phenomenon raises the following questions:

(1) Why does an increase in the variance of the spot rate, which usually indicates a higher uncertainty about real production opportunities, result in *higher* expected one period yields in the future if the length of the period is chosen to be 0.01?

(2) Why does the length of the period seem to influence this result in the one or the other direction?

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Rechts- und Wirtschaftswissenschaftliche Fakultät
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