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Products of non additive measures: A Fubini theorem

Christian Bauer

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Adresse:
Christian Bauer
Universität Bayreuth
Lehrstuhl für Wirtschaftspolitik (VWL I)
95440 Bayreuth

Telefon: +49 - 921 - 552913
Fax: +49 - 921 - 552949
e-mail: Christian.Bauer@uni-bayreuth.de

Abstract

The concept of qualitative differences in information, i.e. the distinction between risk and ambiguity, builds the framework of a growing strand of economic research. For non additive set functions as used in the Choquet Expected Utility framework, the independent product in general is not unique and the Fubini theorem is restricted to slice-comonotonic functions. In this paper, we use the representation theorem of Gilboa and Schmeidler (1995) to extend the Möbius product for non additive set functions to non finite spaces. The uniqueness result of Ghirardato (1997) for belief functions is also extended to non finite spaces. For this unique product, one side of the Fubini theorem holds for all integrable functions if one of the marginals either is a probability or a convex combination of a chain of unanimity games.

JEL: D81, D84

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1 Introduction

The concept of qualitative differences in information, i.e. the distinction between risk and ambiguity introduced by Knight (1921), builds the framework of a growing strand of economic research (see Ghirardato (1997) for a concise literature overview). Ellsberg (1961) introduces the attitude of "uncertainty aversion" in his famous paradox. Choquet Expected Utility framework is presented among many others in Eichberger and Kelsey (1999) or in Dow and da Costa Werlang (1992), who show that under ambiguity inaction may be an optimal investment decision. A growing strand of literature is devoted to the aspects of strategic uncertainty. Coordination games used to model the economics of partnerships or currency crises are always faced with the problem strategic uncertainty, especially if multiple equilibria arise. Spanjers and Kelsey (2004) address this problem for partnerships, while Spanjers (1998/2005) and Bauer (2005) analyze the effects of strategic uncertainty on currency crises. Most of this work, however, is restricted to the univariate case lacking natural extensions to portfolio theory or the inclusion of additional sources of uncertainty. So far literature lacks a formalism for products of non additive measures on continuous spaces and the present paper aims to contribute to closing this gap.

On the technical side, the distinction between risk and ambiguity is a matter of the additivity of the set functions which represent the information set of the decision makers. While risk is modeled by a probability measure, i.e. a σ-additive set function, ambiguity is represented by set functions, which are in general not additive. This imposes some restrictions to the applicability of several techniques which are common for the decision theory under risk. In particular, the Choquet integral is not linear with respect to the integrand, conditioning on new information (updating, learning) cannot be performed in the classical way (see e.g. Denneberg (2002)), and the Fubini theorem does not hold in the classical sense. In general, there is more than one independent product associated with a pair of non additive set functions (see e.g. Walley and Fine (1982), Gilboa and Schmeidler (1989), Hendon et al.

(1996), Koshevoy (1998), and Denneberg (2000, 2002) for alternative approaches) and the iterated Choquet integrals do not equal the integral w.r.t. the proposed product. The seminal article of Ghirardato (1997) addresses these problems. Key element of his approach is the comonotonity properties of functions and sets (see definition 4 below). The order of integration is interchangeable for every pair of marginal set functions if and only if the integrand is slice-comonotonic. And for any not Fubini independent product, there is a slice-comonotonic function for which the integral w.r.t. the product does not equal the iterated integral w.r.t. the marginals. A product is Fubini independent if and only if the latter equality holds for the indicator function of all comonotone sets (see definition 5 below).

Finally, Ghirardato (1997) considers the special case of belief functions on finite spaces. For such two marginals, Theorem 3 states that there is only one Fubini independent product. This special product is the Möbius product.

In this paper, we use the representation theorem of Gilboa and Schmeidler (1995) to generalize the definition the Möbius product for non additive set functions to non finite spaces and show a number of its properties. Firstly, the Möbius product of any capacity with a belief function is a Fubini independent capacity. Secondly, the uniqueness result of Ghirardato (1997) for two marginal belief functions is also extended to non finite spaces. Thirdly, for this unique product, we show, that the integral w.r.t. the product equals the iterated integral w.r.t. the marginals in a certain order for all integrable functions if one of the marginals either is a probability or a convex combination of a chain of unanimity games. As a rule of thumb, the inner integral should be the more ambiguous one.

The remainder of the paper is organized as follows. Section 2 presents the mathematical preliminaries, i.e. definition, notation and main results of Gilboa and Schmeidler (1995), on which this paper is based on, in particular the Möbius representation of capacities on infinite spaces. Section 3 gives an introduction the issue of products for non additive set functions with an explicit referral to the results of Ghirardato (1997). Section 4 then presents the innovative part of the paper. The Möbius product is extended to infinite spaces and its properties are derived. Section 5 concludes.

2 Mathematical preliminaries

Capacities and the Choquet integral are mainly based on the original work of Choquet (1953) and the axiomatization in Schmeidler (1986). They are not only an important tool in game and decision theory models, but also in a wide range of other scientific fields from artificial intelligence to statistics. Thus there is a wide range of notations and terminology. This paper draws on the definitions and notation in Gilboa and Schmeidler (1995), albeit we concentrate on normalized and monotone set functions.

Definition 1 *Let Ω be a non empty set of states of the world, Σ a σ-algebra in Ω and ψ a set function on (Ω, Σ) with $\psi(\emptyset) = 0$.*

1. *ψ is* monotone, *if $A \subseteq B$ implies $\psi(A) \leq \psi(B)$ for all $A, B \in \Sigma$.*

2. *ψ is* normalized, *if $\psi(\Omega) = 1$.*

3. *ψ is called a* capacity, *if it is normalized and monotone. The set of all capacities on (Ω, Σ) is denoted by $\mathcal{K}(\Omega, \Sigma)$ or $\mathcal{K}$.*

4. *ψ is* convex *(supermodular, 2-monotone), if $\psi(A \cap B) + \psi(A \cup B) \geq \psi(A) + \psi(B)$ for all $A, B \in \Sigma$.*

5. *ψ is* totally monotone, *if for any finite set of events $A_i \subseteq \Sigma$, $i \in I$*

$$\psi\left(\bigcup_{i\in I} A_i\right) \geq \sum_{\substack{J\subset I \\ J\neq\emptyset}} (-1)^{|J|+1} \psi\left(\bigcap_{j\in J} A_i\right) \tag{1}$$

   A totally monotone capacity is called belief function.

6. *The capacity* u_A, $A \in \Sigma$ *defined by*

$$u_A(B) = \begin{cases} 1 & \text{if} \quad B \supseteq A \\ 0 & \quad \text{otherwise} \end{cases} \tag{2}$$

is called unanimity game *on* A. *Unanimity games are belief functions.*

7. ψ *is* additive, *if* $\psi(A \cup B) = \psi(A) + \psi(B)$ *for all disjoint* $A, B \in \Sigma$.

8. ψ *is* σ-additive, *if* $\psi\left(\bigcup_{i \in I} A_i\right) = \sum_{i \in I} \psi(A_i)$ *for any countable set of events* $A_i \subseteq \Sigma$, $i \in I$, $\bigcup_{i \in I} A_i \in \Sigma$, $A_i \cap A_j = \emptyset$ *for* $i \neq j$.

9. *The dual capacity* $\overline{\psi}$ *to any capacity* ψ *is given by* $\overline{\psi}(A) = 1 - \psi(A^c)$. *For any set* $\mathcal{M}$ *of capacities the set of dual capacities is denoted by* $\mathcal{M}^D = \left\{\overline{\psi} : \psi \in \mathcal{M}\right\}$.

Unanimity games form a linear basis for all capacities and thus play a key role comparable to the unit vectors in linear algebra.

Definition 2 *For any space* (Ω, Σ), *we define* $\Sigma^- = \Sigma \setminus \emptyset$ *and* $\Psi = \Psi(\Omega, \Sigma)$ *is the algebra generated by the set* $\Theta = \left\{\widetilde{A} : A \in \Sigma^-\right\}$, *where* $\widetilde{A} = \{B \in \Sigma^- : B \subseteq A\}$.

Theorem 1 *Gilboa and Schmeidler (1995, Theorem A) give a general Möbius representation theorem for capacities based on the unanimity games.*

$$\psi(\cdot) = \int_{\Sigma^-} u_T(\cdot)\, d\mu_\Sigma^\psi(T), \tag{3}$$

where μ_Σ^ψ *is an additive measure on the space* (Σ^-, Ψ). *A capacity is a belief function, if and only if* μ_Σ^ψ *is nonnegative and each nonnegative measure* μ *on* Σ *defines a belief function.*

With the above notation we have $\psi(A) = \mu_\Sigma^\psi\left(\widetilde{A}\right)$.

Thus, E-capacities (see Eichberger and Kelsey (1999)), i.e. a convex combination of an additive probability π and unanimity games u_{A_i}, are belief functions, since $\mu(\psi) =$

$\mu\left(\lambda\pi+\sum_{i=1}^{\infty}\lambda_i u_{A_i}\right) = \lambda\mu(\pi) + \sum_{i=1}^{\infty}\lambda_i\mu(u_{A_i})$ is nonnegative. E-capacities are predestined to combine two different kinds of possible situations, a decision under risk, in which probability assumptions are made, and a decision under ambiguity, in which the available information is not sufficient to form a probability.

As a corollary to theorems C and D in Gilboa and Schmeidler (1995), we know that for ψ a finite polynomial of σ-additive measures and unanimity games, the Möbius representation μ_Σ^ψ has an unique σ-additive extension to the σ-algebra generated by Ψ.

Any capacity ψ on a finite Ω may be written as

$$\psi = \sum_{A\in P(\Omega)} \varphi_\psi(A)u_A \tag{4}$$

This notation is called Möbius representation. The Möbius coefficients $\varphi_\psi(A)$ may be calculated as $\varphi_\psi(A) = \psi(A) - \sum_{J\subseteq\{1,..,n\},\ J\neq\emptyset}(-1)^{|J|+1}\psi(\cap_{j\in J}A_i)$ with $A_j = A\backslash\{\omega_j\}$ and $A=\{\omega_1,...,\omega_n\}$.

A capacity is a belief function, if and only if all its Möbius coefficients are nonnegative. In this case, $\varphi_\psi(A)$ may be viewed as the amount of information that cannot be distributed to further subsets from A. The positivity of the Möbius transformation of belief functions will be especially useful in the next section. It allows to receive results for products of capacities where at least one of the capacities is a belief function.

Notation 1 *To simplify notation the mapping, which assigns the corresponding additive measure on the space* Σ^- *to each capacity on* (Ω,Σ)*, is denoted by* μ*, i.e.* $\mu:\psi\longmapsto\mu_\Sigma^\psi$ *or* $\mu(\psi)=\mu_\Sigma^\psi$ *or, if there is no misinterpretation possible,* μ^ψ*.*

The mapping μ is linear, i.e. $\mu(\lambda\psi_1+(1-\lambda)\psi_2)=\lambda\mu(\psi_1)+(1-\lambda)\mu(\psi_2)$.

Example 1 *1.* $\mu(u_A)(\{T\}) = \begin{cases} 1 & \text{if } T=A \\ 0 & \text{otherwise}\end{cases} = 1_{\{A\}}(\{T\})$.

2. If Σ *is finite, then* μ^ψ *is a simple discrete measure with* $\mu^\psi(T)=\varphi_\psi(T)$*.*

3. *If ψ is an additive probability on Ω, then $\mu^{\psi} \equiv 0$ on $\Sigma \setminus \{\{\omega\} : \omega \in \Omega\}$ and $\mu^{\psi}(\{\{\omega\} : \omega \in A\}) = \psi(A)$, i.e. μ^{ψ} is the natural injection of ψ into the space Σ.*

4. *If $\mu^{\psi}(T) > 0$ for some $T \subset \Sigma \setminus \{\{\omega\} : \omega \in \Omega\}$, then ψ is not additive.*

While convex capacities represent pessimistic attitudes, the dual capacities of convex are concave and represent optimistic behavior. Note that the dual capacity to a belief function is not necessarily a plausibility measure, i.e. a capacity with a Möbius transformation that takes only negative values on all non singletons.

The Choquet integral developed by Choquet (1953) and axiomatized by Schmeidler (1986) is commonly used in decision theory to formalize expectations under ambiguity.[1] Note that the Choquet integral is not linear like the Lebesgue-integral, as any integral w.r.t. a non additive set function cannot be linear.

Definition 3 *For any integrable*[2] *function $f : (\Omega, \Sigma) \to (\mathbb{R}, \mathbb{B})$ the* Choquet integral *is defined as*

$$\mathbb{CE}_{\psi}(f) = \int f(\omega) d\psi(\omega) \tag{5a}$$

$$= \int_0^{\infty} \psi\left(\omega \in \Omega : f(\omega) \geq \alpha\right) d\alpha + \int_{-\infty}^{0} \left[\psi\left(\omega \in \Omega : f(\omega) \geq \alpha\right) - 1\right] d\alpha \tag{5b}$$

where the latter integrals are in Lebesgue sense. If ψ is an additive probability the Choquet integral equals the Lebesgue one.[3]

Lemma 2 *Gilboa and Schmeidler (1995) show that the Choquet-expectation of a capacity ψ on (Ω, Σ) can be expressed as*

$$\mathbb{CE}_{\psi}(f) = \int\limits_{T \in \Sigma^{-}} \inf_{\omega \in T} f(\omega) d\mu(\psi). \tag{6}$$

[1] An alternative is the Sugeno integral which is widely used in fuzzy theory. In some sense, this situation is comparable to the theory of stochastic calculus, having e.g. the Ito integral and the Strantonowich integral.

[2] The term integrable means that the expression (5b) is well defined, i.e. it is not the case that both $f^{+} = \max(0, f)$ and $f^{-} = -\min(0, f)$ have an infinite expectation.

[3] Properties of the Choquet integral are given in the appendix.

3 Products

Most applications examine functions of more than one variable. E.g. the return of a portfolio consisting of different assets is a weighted sum of the returns of the single assets. To work with more than one variable one needs to define product and independence. For nonadditive set functions, however, the independent product is not unique and the Fubini theorem does not hold for all functions (see e.g. Walley and Fine (1982), Gilboa and Schmeidler (1989), Hendon et al. (1996), Koshevoy (1998), and Denneberg (2000) for alternative approaches for a product of monotone measures. Denneberg (2002) defines a product using the max–min additive representation of monotone measures, which coincides for supermodular set functions with the methods of Walley and Fine (1982), Gilboa and Schmeidler (1989), as well as for totally monotone measures with the Möbius product in Hendon et al. (1996), Ghirardato (1997), and the present paper.

Notation 2 *For the remainder of the paper ψ_X and ψ_Y denote two capacities with variables X and Y on finite dimensional real spaces (Ω_X, Σ_X) resp. (Ω_Y, Σ_Y). $\mu(\psi_X)$ and $\mu(\psi_Y)$ denote their respective Möbius representations on (Σ_X^-, Ψ_X) and (Σ_Y^-, Ψ_Y). $(\Omega_{X\times Y}, \Sigma_{X\times Y})$ is the product space with the induced σ-algebra. Möbius representations of capacities on $\Omega_{X\times Y}$ live on $\left(\Sigma_{X\times Y}^-, \Psi_{X\times Y}\right)$, where $\Psi_{X\times Y} = \Psi\left(\Sigma_{X\times Y}^-\right)$ is defined as in definition 2.*

Remark 1 *It is very important to keep in mind the implications of this hierarchy of spaces. While in the original level $X \times Y$ is the direct product of X and Y and the σ-algebra of the product space $\Sigma_{X\times Y}$ is the σ-algebra induced by the σ-algebras of the marginals, i.e. $\Sigma_{X\times Y} = \sigma(\Sigma_X \times \Sigma_Y)$. This does not transfer to the level of the σ-algebra spaces.*
The direct product $\Sigma_X \times \Sigma_Y$ only consists of all rectangles in $X \times Y$ and includes but does not equal the induced σ-algebra $\Sigma_{X\times Y}$ on the product space, i.e. $\Sigma_{X\times Y} \supsetneqq \Sigma_X \times \Sigma_Y$. This implies $\Psi(\Sigma_{X\times Y}) \supsetneqq \Psi(\Sigma_X \times \Sigma_Y)$. Therefore, $\left(\Sigma_{X\times Y}^-, \Psi_{X\times Y}\right)$ is not *the product space of $\left(\Sigma_X^-, \Psi_X\right)$ and $\left(\Sigma_Y^-, \Psi_Y\right)$.*

Let ψ_X and ψ_Y be two capacities with joint distribution $\psi_{X\times Y}$. They are independent, if

$$\psi_{X\times Y}(A\times B)=\psi_X(A)\psi_Y(B)\ \forall A\in\Sigma_X, B\in\Sigma_Y. \tag{7}$$

For σ-additive probabilities this defines an unique $\sigma-$additive measure on $\Omega_{X\times Y}$. If either ψ_X or ψ_Y are not additive, in general there is more than one product measure satisfying (7).

Example 2 *Let $\Omega_X = \Omega_Y = \{0,1\}$ and $\psi_X = \psi_Y = u_{\{0,1\}}$. Then there are several independent measures on the product space $\Omega_{X\times Y}$. Two examples are $\psi_1 = u_{\Omega_X\times\Omega_Y}$ or $\psi_2 = u_{\{(0,0),(1,1)\}}$, i.e. the unanimity game of the whole product space resp. the unanimity game on the diagonal. It is easy to verify, that (7) holds for ψ_1 and ψ_2 since all rectangles except for the whole space have measure 0. For additive measures the measure of the diagonal would be uniquely determined as the sum of the measures of its elements.*

For independent variables with additive probabilities π_X and π_Y one has the Fubini theorem for integrable functions

$$\begin{aligned}\int f(x,y)d\pi_{X\times Y} &= \int\left(\int f(x,y)d\pi_X\right)d\pi_Y \\ &= \int\left(\int f(x,y)d\pi_Y\right)d\pi_X,\end{aligned} \tag{8}$$

i.e. the order of integrations does not matter. For nonadditive set functions the Fubini theorem does not hold for all functions.

Ghirardato (1997) tackles both questions, i.e. (1) "Is there a sensible way of uniquely defining an independent product of non additive set functions?" and (2) "When are the iterated integrals w.r.t. the marginals equal and when do they equal the integral w.r.t. the proposed product?". The definitions below are the key elements of Ghirardato (1997) answers.

Definition 4 *Definition 3 in Ghirardato (1997):*

1. *Two real valued functions f and g are* comonotone, *if $\forall x,\ x'$*

$$(f(x)-f(x'))\,(g(x)-g(x'))\geq 0. \tag{9}$$

2. *A function* $f : \Omega_X \times \Omega_Y \to \mathbb{R}$ *is called* slice-comonotone*, if* $\forall x,\ x' \in \Omega_X$ *the functions* $f(x,\cdot)$ *and* $f(x',\cdot)$ *are comonotone. The functions* $f(\cdot,y)$ *and* $f(\cdot,y')$ *then are comonotone too.*

3. *A set* $C \subset \Omega_X \times \Omega_Y$ *is called* comonotone*, if its indicator function* 1_C *is slice-comonotone.*

Definition 5 *Definition 5 in Ghirardato (1997): The set of product capacities on* $\Omega_{X\times Y}$ *that satisfies (7) is denoted by* $\mathcal{P}(\psi_X, \psi_Y)$ *or* $\mathcal{P}$*. The set of product capacities that in addition has the Fubini property, i.e.* $\forall$ *comonotone* $C \subset \Omega_X \times \Omega_Y$

$$\psi_{X\times Y}(C) = \mathbb{CE}_{\psi_X}\left(\mathbb{CE}_{\psi_Y}(1_C)\right) = \mathbb{CE}_{\psi_Y}\left(\mathbb{CE}_{\psi_X}(1_C)\right) \tag{10}$$

is denoted by $\mathcal{F}(\psi_X, \psi_Y)$ *or* $\mathcal{F}$*. A capacity* $\psi \in \mathcal{F}$ *is called* Fubini independent product*.*

In particular, Ghirardato (1997) shows that for all product capacities, which have the Fubini property, the Fubini theorem holds for all slice-comonotonic functions. The order of integration is interchangeable for every pair of marginal set functions if and only if the integrand is slice-comonotonic. And for any not Fubini independent product, there is a slice-comonotonic function for which the integral w.r.t. the product does not equal the iterated integral w.r.t. the marginals.

Finally, for two belief functions on finite spaces, his Theorem 3 states that there is a unique Fubini independent product. This special product is the Möbius product. Using the representation theorem of Gilboa and Schmeidler (1995), we generalizes the Möbius product and this result in the next section.

4 The Möbius-Product and a Fubini-like theorem

This section contains the innovation of the paper. The Möbius product, which we will introduce below, generalizes approach introduced Ghirardato (1997). In detail, we proceed as follows: Lemma 3 defines the Möbius product as a set function and proves an important integral equation. The Möbius product for non additive measures, which is defined for finite spaces in definition 7 in Ghirardato (1997), is extended to continuous spaces. Theorem 4 shows that the Möbius product is a capacity if one of the marginals is a belief function. Theorem 5 extends the uniqueness result of Theorem 3 in Ghirardato (1997) to not finite spaces. Finally, the corollaries 3 and 4 show that for two border cases, i.e. if one of the marginals is either an linear combination of a chain of unanimity games or a probability measure, one half of the Fubini-theorem holds for all integrable functions.

Lemma 3 *Let* $\Psi\left(\Sigma_X^- \times \Sigma_Y^-\right)$ *denote the algebra generated by the direct product of of the spaces* Σ_X^- *and* Σ_Y^- *and* $\mu\left(\psi_X\right) \otimes \mu\left(\psi_Y\right)$ *the additive product measure of* $\mu\left(\psi_X\right)$ *and* $\mu\left(\psi_Y\right)$ *on* $\left(\Sigma_X^- \times \Sigma_Y^-, \Psi\left(\Sigma_X^- \times \Sigma_Y^-\right)\right)$. *Then* $\mu\left(\psi_X\right) \otimes \mu\left(\psi_Y\right)$ *has an additive extension to* $\left(\Sigma_{X\times Y}^-, \Psi_{X\times Y}\right)$, *where* $\Psi_{X\times Y} = \Psi\left(\Sigma_{X\times Y}^-\right)$. *Denote this extended product by* $\mu\left(\psi_X\right) \overset{e}{\otimes} \mu\left(\psi_Y\right)$. *Furthermore*

$$\int\limits_{T_X \times T_Y \in \Sigma_X^- \times \Sigma_Y^-} u_{T_X \times T_Y}\left(A\right) d\left[\mu\left(\psi_X\right) \overset{e}{\otimes} \mu\left(\psi_Y\right)\right]\left(T_X \times T_Y\right) \tag{11}$$

$$= \int\limits_{T_X \in \Sigma_X^-} \left(\int_{T_Y \in \Sigma_Y^-} \inf_{(x,y) \in T_X \times T_Y} 1_A(x,y) d\mu^{\psi_Y}\left(T\right) \right) d\mu^{\psi_X}\left(T\right) \tag{12}$$

holds for all $A \in \Sigma_{X\times Y}$.

Definition 6 *The* independent Möbius product *is given by*

$$\psi_X \overset{M}{\otimes} \psi_Y\left(A\right) = \int\limits_{T_X \times T_Y \in \Sigma_X^- \times \Sigma_Y^-} u_{T_X \times T_Y}\left(A\right) d\left[\mu\left(\psi_X\right) \overset{e}{\otimes} \mu\left(\psi_Y\right)\right]\left(T_X \times T_Y\right). \tag{13}$$

The figure below gives a graphical representation of the way, we define the Möbius product. The detour using the chain of the direct product and the additive extension is necessary, because $\left(\Sigma_{X\times Y}^-, \Psi_{X\times Y}\right)$ is not the product space of $\left(\Sigma_X^-, \Psi_X\right)$ and $\left(\Sigma_Y^-, \Psi_Y\right)$ (see remark 1).

$$\left(\Sigma_X^- \times \Sigma_Y^-, \Psi\left(\Sigma_X^- \times \Sigma_Y^-\right), \mu\left(\psi_X\right) \otimes \mu\left(\psi_Y\right)\right)$$

$\nearrow$ direct product additive extension $\searrow$

$$\left(\Sigma_X^-, \Psi_X, \mu\left(\psi_X\right)\right) \quad \left(\Sigma_Y^-, \Psi_Y, \mu\left(\psi_Y\right)\right) \qquad\qquad \left(\Sigma_{X\times Y}^-, \Psi_{X\times Y}, \mu\left(\psi_X \overset{e}{\otimes} \psi_Y\right)\right)$$

$$\updownarrow \mathrm{M} \qquad\qquad \updownarrow \mathrm{M} \qquad\qquad\qquad\qquad \updownarrow \mathrm{M}$$

$$\left(X, \Sigma_X, \psi_X\right) \quad \left(Y, \Sigma_Y, \psi_Y\right) \quad \xrightarrow[\text{new definition}]{\text{Möbius Product}} \quad \left(X \times Y, \Sigma_{X\times Y}, \psi_X \overset{M}{\otimes} \psi_Y\right)$$

Figure 1: Graphical representation of definition 6 of the Möbius product

Note that $\mu\left(\psi_X\right) \overset{e}{\otimes} \mu\left(\psi_Y\right)$ is not necessarily positive and that therefore $\psi_X \overset{M}{\otimes} \psi_Y$ is not necessarily monotone, i.e. no capacity, for arbitrary capacities ψ_X and ψ_Y. Nonetheless the integrals in (11) and (12) are well defined. Theorem 4 and corollary 1 give a sufficient condition for the monotonicity of the product: the total monotonicity of one of the marginals. The independent Möbius product derives its name from the extension principle for $\overset{e}{\otimes}$. The additive extension of $\mu\left(\psi_X\right) \otimes \mu\left(\psi_Y\right)$ lays (in the space of the Möbius representation) no weight on the elements of $\Sigma_{X\times Y}$ that are no rectangles.[4] This is the equivalent construction principle of the Möbius product for finite spaces.[5]

Theorem 4 *The Möbius product of a belief function with any other capacity is a Fubini independent product capacity, i.e.*

$$\mu\left(\psi_X\right) \geq 0 \Longrightarrow \psi_X \overset{M}{\otimes} \psi_Y \in \mathcal{F}\left(\psi_X, \psi_Y\right). \tag{14}$$

The proof of this theorem is a simple combination of the two following corollaries.

Corollary 1 *A sufficient condition that ensures the monotonicity of the Möbius product is the positivity of the Möbius transformation of one of the marginal measures, i.e.*

$$\mu\left(\psi_X\right) \geq 0 \Longrightarrow \psi_X \overset{M}{\otimes} \psi_Y\left(A \cup B\right) \geq \psi_X \overset{M}{\otimes} \psi_Y\left(A\right) \forall A, B \subset \Omega_X \times \Omega_Y. \tag{15}$$

[4]Note that naturally the product capacity does not disappear on non rectangles, but that the measure of a set A is given by the weight of the Möbius representation of all rectangles contained in A.

[5]If ψ_X and ψ_Y are σ-additive measures, the mapping μ is the natural injection (see example 3) and the Möbius product coincides with the usual product measure.

Corollary 2 *Eq. (10) holds for the Möbius product of any pair of capacities. Thus any Möbius product that is a capacity is Fubini independent, i.e.*

$$\psi_X \overset{M}{\otimes} \psi_Y \in \mathcal{K}\left(\Omega_X \times \Omega_Y\right) \Longrightarrow \psi_X \overset{M}{\otimes} \psi_Y \in \mathcal{F}\left(\pi_X, \pi_Y\right).$$

If both marginals have a positive Möbius transformation, the existence theorem can be extended to an uniqueness theorem.

Theorem 5 *The Möbius product $\psi_X \overset{M}{\otimes} \psi_Y$ of two belief functions ψ_X and ψ_Y is the only Fubini independent product capacity. Further it safisfys the Fubini theorem for any integrable slice-comonotone function. It is a belief function.*

For finite spaces this simplifies to the known Möbius product formulas, given in e.g. Ghirardato (1997). For additive measures this simplifies to the usual product measure. The theorem may be applied to non bounded functions and extends the results of Ghirardato (1997) with respect to the class of functions as well as to the class of measures. The continuous setup allows to combine capacities and continuous probability distributions like the normal distribution and thus to apply capacities to standard economic theories like portfolio or risk management theory.

Example 3 *1. Let ψ_x be any capacity and $\psi_Y = u_A$ for some event A, then*

$$\psi_X \overset{M}{\otimes} \psi_Y = \psi_X \overset{M}{\otimes} u_A = \begin{cases} \mu(\psi_X) & \text{on } \Sigma_X \times \{A\} \\ 0 & \text{otherwise} \end{cases}.$$

The event $T \subset \Omega_X \times \Omega_Y$ therefore has probability $\psi_X \overset{M}{\otimes} u_A(T) = \psi_X(\{t : (t, A) \subseteq T\})$.

2. Let $\psi_X = \lambda_X \pi_X + (1 - \lambda_X) u_A$ and $\psi_Y = \lambda_Y \pi_Y + (1 - \lambda_Y) u_B$ be two E-capacities, then

$$\begin{aligned} \psi_X \overset{M}{\otimes} \psi_Y &= \lambda_X \lambda_Y \pi_X \otimes \pi_Y \\ &\quad + \lambda_X (1 - \lambda_Y) \pi_X \overset{M}{\otimes} u_B + (1 - \lambda_X) \lambda_Y u_A \overset{M}{\otimes} \pi_Y \\ &\quad + (1 - \lambda_X)(1 - \lambda_Y) u_A \overset{M}{\otimes} u_B \end{aligned}$$

The independent Möbius product has the Fubini property. For any comonotone function f the order of integration does not matter and (8) holds. But even simple problems like the distribution of the sum of two variables involve non comonotone functions.[6] Therefor, the applicability is rather limited.

It is also well known (see e.g. Ghirardato (1997)) that there is no product of nonadditive measures satisfying all equation signs of the Fubini theorem for all functions. The uniqueness part of theorem 5 shows that all products with the Fubini property for comonotone functions must coincide with the Möbius product. The following corollaries show that the Möbius product yields one half of the Fubini theorem for integrable functions, if one of the marginal measures is a probability measure or a convex combination of a chain of unanimity games.

Corollary 3 *Let ψ_X be any capacity and $\psi_Y = \pi$ a σ-additive probability, then for any integrable f on the product space*

$$\int f d\psi_X \overset{M}{\otimes} \pi = \int \left(\int f d\psi_X \right) d\pi \tag{16}$$

holds. The reverse order of integration in general yields a different result.

Corollary 4 *Let $\psi_X = \sum_{i=1}^n \lambda_i u_{A_i}$, $\sum_{i=1}^n \lambda_i = 1$, $A_i \in \Sigma \forall i$, $i < j \Longrightarrow A_i \subseteq A_j$ and ψ_Y be any capacity, then for any measurable index function 1_B on the product space*

$$\int 1_B d\psi_X \overset{M}{\otimes} \psi_Y = \int \left(\int 1_B d\psi_X \right) d\psi_Y \tag{17}$$

holds and for any measurable function f that is bounded below

$$\int f d\psi_X \overset{M}{\otimes} \psi_Y = \int \left(\int f d\psi_X \right) d\psi_Y = \mathbb{CE}_{\psi_Y} \left(\sum_{i=1}^n \lambda_i \inf_{x \in A_i} f(x,y) \right). \tag{18}$$

If ψ_Y is an E-capacity (see Eichberger and Kelsey (1999)), i.e. a convex combination of an additive probability π and unanimity games u_{A_i}, $\psi_Y = \lambda\pi + \sum_{i=1}^n \lambda_i u_{A_i}$, $\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = 1 - \lambda$, $A_i \in \Sigma \forall i$, eq. (18) holds for any integrable function f.

[6] The indicator function of the event $X+Y \in A$ is not comonotone, since $X+Y \in A$ is a diagonal and thus not a comonotone set. Thus the Fubini theorem cannot be used to calculate the probability of $X+Y \in A$ if the distributions of X and Y are nonadditive.

Example 4 *Distribution of a sum: Let* $X \sim \pi$*, i.e. a random variable with a* σ*-additive probability distribution* π*, and* $Y \sim u_{[-a,a]}$*, i.e. a random variable with values in* $[a, \widetilde{a}]$*. If* X *and* Y *are independent, then the probability for* $X+Y$ *to lie in an interval* $[b, \widetilde{b}]$ *is* 0 *if* $\widetilde{b} - b < \widetilde{a} - a$*, i.e. there is no* $x \in \mathbb{R}$ *with* $[b+x, \widetilde{b}+x] \supseteq [a, \widetilde{a}]$*, otherwise*

$$\begin{aligned} \mathbb{P}(X+Y \in [b, \widetilde{b}]) &= \int 1_{[b,\widetilde{b}]}(x+y) d\left(u_{[a,\widetilde{a}]} \overset{M}{\otimes} \pi\right) \\ &= \pi(\{x : [b-x, \widetilde{b}-x] \supseteq [a, \widetilde{a}]\}) \\ &= \pi([b-a, \widetilde{b}-\widetilde{a}]) \end{aligned}$$

Corollary 5 *For any pair of capacities* ψ_X *and* ψ_Y

$$\mathcal{F}\left(\psi_X, \psi_Y\right)^D = \mathcal{F}\left(\overline{\psi_X}, \overline{\psi_Y}\right) \tag{19}$$

Corollary 6

$$\mathcal{F}\left(\overline{u_A}, \overline{u_B}\right) = \left\{\overline{u_A \overset{M}{\otimes} u_B}\right\} \tag{20}$$

5 Conclusion

In this paper, we have used the representation theorem of Gilboa and Schmeidler (1995) to generalize the definition the Möbius product for non additive set functions to non finite spaces. The Möbius product offers a number of comfortable properties, which might assign it a preferential place among the possible products of non additive set functions. Firstly, the Möbius product of any capacity with a belief function is a Fubini independent capacity. Secondly, for two belief functions, it is the only Fubini independent product. Thirdly, for this unique product, the integral w.r.t. the product equals the iterated integral w.r.t. the marginals in a certain order for *all integrable functions* if one of the marginals either is a probability or a convex combination of a chain of unanimity games. As a rule of thumb, the inner integral should be the more ambiguous one.

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APPENDIX

APPENDIX A: List of Notations

ψ	capacity
π	additive probability
μ	representational mapping for capacities (p. 6)
Ω	space of possible states
Σ	sigma algebra on Ω
Σ^-	$\Sigma \setminus \emptyset$, sigma algebra on Ω without the empty set
$\times$	Cartesian product of sets $(A \times B)$
$\otimes$	space on the Cartesian product of sets $(\Sigma_X \otimes \Sigma_Y)$
$\otimes$	independent product of additive measures $(\mu(\psi_X) \otimes \mu(\psi_Y))$
$\overset{M}{\otimes}$	independent Möbius product of non additive set functions $(\psi_X \overset{M}{\otimes} \psi_Y)$ (p. 5)
u_A	unanimity game for the set A (p. 2)

APPENDIX B: Proofs from chapter 4

Proof 1 *Proof of lemma 3*

The proof of the extension is a two-stepped back and forth between the original product space $\Omega_X \times \Omega_Y$ and the "product space" in the Möbius representation $\Sigma_X^- \times \Sigma_Y^-$ respectively $\Sigma_{X\times Y}^-$. The main difficulty is, that the product of the spaces on which the Möbius transformations live is smaller than the space on which the Möbius transformation of the product lives, i.e. $\Sigma_X^- \times \Sigma_Y^- \subsetneqq \Sigma_{X\times Y}^-$. Therefore, even if we had the case that $\mu(\psi_X)$ and $\mu(\psi_Y)$ are σ-additive and thus is their product, we could not use the classical Fubini theorem (see also remark 1) First a set function ψ on $\Omega_X \times \Omega_Y$ is defined by

$$\psi(B) = \sup_{A \subset B, A \in \alpha\left(\Sigma_X^- \times \Sigma_Y^-\right)} \int_{T_X \times T_Y \in \Sigma_X^- \times \Sigma_Y^-} u_{T_X \times T_Y}(A) \, d\left[\mu(\psi_X) \otimes \mu(\psi_Y)\right](T_X \times T_Y) \qquad (21)$$

for $B \in \Sigma_{X\times Y}$, where $\alpha\left(\Sigma_X^- \times \Sigma_Y^-\right)$ denotes the algebra generated by $\Sigma_X^- \times \Sigma_Y^-$. We denote *the Möbius representation* $\mu(\psi)$ *of* ψ *by* $\mu(\psi_X) \overset{e}{\otimes} \mu(\psi_Y)$. *Now* $\mu(\psi_X) \overset{e}{\otimes} \mu(\psi_Y)$ *(defined on* $\Psi\left(\Sigma^-_{X\times Y}\right)$*) is the additive extension of* $\mu(\psi_X) \otimes \mu(\psi_Y)$ *(defined on* $\Psi\left(\Sigma^-_X \times \Sigma^-_Y\right)$*)* . *Due to theorem 1* $\mu(\psi)$ *is an additive measure on* $\left(\Sigma^-_{X\times Y}, \Psi_{X\times Y}\right)$ *by definition. Also for* $A \in \alpha\left(\Sigma^-_X \times \Sigma^-_Y\right)$ *the supremum takes its value at* A *due to the monotonicity of the integral.*[7] *Thus*

$$\mu(\psi_X) \overset{e}{\otimes} \mu(\psi_Y) |_{\left(\Sigma^-_X\times\Sigma^-_Y, \Psi\left(\Sigma^-_X\times\Sigma^-_Y\right)\right)} \equiv \mu(\psi_X) \otimes \mu(\psi_Y). \tag{22}$$

At last (11) is proved. For $A \in \Sigma_{X\times Y}$

$$\int\limits_{T_X\times T_Y\in\Sigma^-_X\times\Sigma^-_Y} u_{T_X\times T_Y}(A)\, d\left[\mu(\psi_X) \overset{e}{\otimes} \mu(\psi_Y)\right](T_X \times T_Y)$$
$$= \int\limits_{T_X\times T_Y\in\Sigma^-_X\times\Sigma^-_Y} u_{T_X\times T_Y}(A)\, d\left[\mu(\psi_X) \otimes \mu(\psi_Y)\right](T_X \times T_Y)$$
$$= \int\limits_{T_X\in\Sigma^-_X} \left(\int_{T_Y\in\Sigma^-_Y} \inf_{(x,y)\in T_X\times T_Y} 1_A(x,y) d\mu^{\psi_Y}(T_Y)\right) d\mu^{\psi_X}(T_X).$$

The first equation sign holds due to (22), since both measures are identical over the area of integration $\Sigma^-_X \times \Sigma^-_Y$. *The second equation sign holds since* $u_{T_X\times T_Y}(A) = \inf_{(x,y)\in T_X\times T_Y} 1_A(x,y)$ *and is* Ψ_Y*-measurable for given* $T_X \in \Sigma^-_X$ *since* $u_{T_X\times T_Y}(A) = 1_{P_Y((T_X\times\Omega_Y)\cap A)}(T_Y)$ *and* $P_Y((T_X \times \Omega_Y) \cap A) \in \Sigma_Y$, *where* P_Y *denotes the projection to* Ω_Y.

[7] Each integral $\int d\mu(\psi_X)$ resp. $\int d\mu(\psi_Y)$ is monotone since ψ_X and ψ_Y are monotone. Thus integral on the product measure $\mu(\psi_X) \otimes \mu(\psi_Y)$ is monotone, too, even though the individual Möbius transforms $\mu(\psi_X)$, $\mu(\psi_Y)$, and $\mu(\psi_X) \otimes \mu(\psi_Y)$ are not necessarily positive.

Proof 2 *Proof of corollary 1*

For any set $A \in \Sigma_{X\times Y}$ in the product space

$$\begin{aligned}\psi_X \overset{M}{\otimes} \psi_Y (A) &= \int\limits_{S\times T\in\Sigma_X^- \times \Sigma_Y^-} \inf_{(x,y)\in S\times T} 1_A(x,y) d\left[\mu(\psi_X)\otimes\mu(\psi_Y)\right](S\times T) \\ &= \int\limits_{S\in\Sigma_X^-} \left(\int_{T\in\Sigma_Y^-} \inf_{(x,y)\in S\times T} 1_A(x,y) d\mu^{\psi_Y}(T)\right) d\mu^{\psi_X}(S) \\ &= \int\limits_{S\in\Sigma_X^-} \psi_Y(\{y : S\subset A_y\})\, d\mu^{\psi_X}(S)\end{aligned}$$

holds. Analogously

$$\psi_X \overset{M}{\otimes} \psi_Y (A\cup B) = \int\limits_{S\in\Sigma_X^-} \psi_Y\left(\left\{y : S\subset (A\cup B)_y\right\}\right) d\mu^{\psi_X}(S)$$

can be shown for $B \in \Sigma_{X\times Y}$. Since ψ_Y is monotone for any $S \in \Sigma_X^-$

$$\psi_Y\left(\left\{y : S\subset (A\cup B)_y\right\}\right) \geq \psi_Y(\{y : S\subset A_y\})$$

holds and monotonicity of the integral w.r.t. ψ_X, i.e. $\mu^{\psi_X} \geq 0$, implies $\psi_X \overset{M}{\otimes} \psi_Y (A\cup B) \geq \psi_X \overset{M}{\otimes} \psi_Y (A)$.

Proof 3 *Proof of corollary 2*

Recall that $\mu\left(\psi_X \overset{M}{\otimes} \psi_Y\right)$ vanishes on all non rectangles in $\Sigma_{X\times Y}$, and note that an indictor function of a set A is comonotone, if and only if all its x-sections $A_x = \{y : (x,y) \in A\}$ are ordered by inclusion. For a given set $S \in \Sigma_X$ therefore $\mu^{\psi_Y}(T : T \subseteq A_x \forall x \in S) =$

$\inf_{x\in S} \mu^{\psi_Y}(T : T \subseteq A_x)$ *holds.*[8]

$$\begin{aligned}
\mathbb{CE}_{\psi_X \overset{M}{\otimes} \psi_Y}(1_A) &= \int\limits_{S\in\Sigma_X^-} \left(\int_{T\in\Sigma_Y^-} \inf_{(x,y)\in S\times T} 1_A(x,y) d\mu^{\psi_Y}(T) \right) d\mu^{\psi_X}(S) \\
&= \int\limits_{S\in\Sigma_X^-} \mu^{\psi_Y}(T : T \subseteq A_x \forall x \in S)\, d\mu^{\psi_X}(S) \\
&= \int\limits_{S\in\Sigma_X^-} \inf_{x\in S} \mu^{\psi_Y}(T : T \subseteq A_x)\, d\mu^{\psi_X}(S) \\
&= \int\limits_{S\in\Sigma_X^-} \inf_{x\in S} \left(\int_{T\in\Sigma_Y} \inf_{y\in T} 1_A(x,y) d\mu^{\psi_Y}(T) \right) d\mu^{\psi_X}(S) \\
&= \mathbb{CE}_{\psi_X}\left(\mathbb{CE}_{\psi_Y}(1_A)\right).
\end{aligned}$$

Proof 4 *Proof of theorem 5*

Existence *The product* $\overset{M}{\otimes}$ *is well defined by (13).* $\psi_X \overset{M}{\otimes} \psi_Y$ *is a belief function, since* ψ_X *and* ψ_Y *are belief functions. This is equivalent to* $\mu(\psi_X) \geq 0$ *and* $\mu(\psi_Y) \geq 0$*. Therefore* $\mu\left(\psi_X \overset{M}{\otimes} \psi_Y\right) \geq 0$ *or, in other words,* $\psi_X \overset{M}{\otimes} \psi_Y$ *is a belief function.*

Uniqueness *To prove uniqueness it is shown by contradiction that the Möbius representations (cf. (3)) of any two products with the Fubini property coincide. It is sufficient to show that they do not lay mass on non rectangles. Let* $\psi = \psi_X \overset{M}{\otimes} \psi_Y$ *and* $\widetilde{\psi}$ *another product with the Fubini property. Suppose there is a non rectangle* A *with* $\mu\left(\widetilde{\psi}\right)(A) = \varepsilon > 0$*. Let* C *be a comonotone hull of* A*, i.e. a comonotone set containing* A *and for any other comonotone set* B *yields:* $A \subseteq B \subseteq C \Longrightarrow B = C$*.*[9] *Note that* C *is not a rectangle, since* A *is not a rectangle. The Fubini property gives us*

$$0 < \varepsilon \leq \alpha \overset{def}{=} \psi(C) = \widetilde{\psi}(C). \tag{23}$$

[8] To keep the notation simple the notation μ^{ψ} for $\mu(\psi)$ and $\Sigma^- = \Sigma \setminus \emptyset$ is used for the reminder of the proof.

[9] The comonotone hull of a set exists, but is not unique. For the diagonal one has e.g. the upper and the lower triangle.

There is an x_0 in the x-projection of C with $\psi(C_{x_0}) < \frac{\varepsilon}{2}$, where $C_{x_0} = \{y : (x_0, y) \in C\}$ is the x_0-slice of C. Otherwise $x_1, ..., x_n$ ($x_i \neq x_j \forall i, j$, i.e. $C_{x_i} \cap C_{x_j} = \emptyset \forall i, j$) would exist with $\psi(C_{x_i}) \geq \frac{\varepsilon}{2}$ and $\alpha = \psi(C) = \psi(\bigcup_{i=1}^n C_{x_i}) = \sum_{i=1}^n \psi(C_{x_i})$. Since all C_{x_i} are rectangles $\psi(C_{x_i}) = \widetilde{\psi}(C_{x_i})$ then would imply the contradiction $\alpha = \widetilde{\psi}(C) \geq \mu\left(\widetilde{\psi}\right)(A) + \sum_{i=1}^n \widetilde{\psi}(C_{x_i}) = \varepsilon + \alpha$.

$$\psi(C \setminus C_{x_0}) > \alpha - \frac{\varepsilon}{2}, \text{ but} \tag{24a}$$

$$\widetilde{\psi}(C \setminus C_{x_0}) \leq \alpha - \varepsilon, \tag{24b}$$

holds since $A \nsubseteq C \setminus C_{x_0}$. All x-sections of $C \setminus C_{x_0}$ can be ordered by inclusion, since the x-sections of C can. Therefore $C \setminus C_{x_0}$ is comonotone and

$$\psi(C \setminus C_{x_0}) = \widetilde{\psi}(C \setminus C_{x_0}) \tag{25}$$

holds. But (25) contradicts (24a) and (24b). Thus $\mu\left(\widetilde{\psi}\right)$ does not put mass on non rectangles.

Fubini-Property *The irrelevance of the order of integration for indicator functions is given by corollary 2, i.e.*

$$\psi_{X\times Y}(C) = \mathbb{CE}_{\psi_X}(\mathbb{CE}_{\psi_Y}(1_C)) = \mathbb{CE}_{\psi_Y}(\mathbb{CE}_{\psi_X}(1_C)) \tag{26}$$

$\forall$ comonotone $C \subset \Omega_X \times \Omega_Y$. In the next steps the class of functions is enlarged step-by-step. Any bounded simple comonotone function f can be written as $f = \sum_{i=1}^n \alpha_i 1_{A_i}$ with $\{A_i\}_{i\leq n}$ a chain of comonotone sets. Comonotone additivity generalizes the proof to this class. A simple nonnegative comonotone function f can be written as $f = \sum_{i=1}^\infty \alpha_i 1_{A_i}$ with $\alpha_i \geq 0$ and $\{A_i\}_{i\leq n}$ a chain of comonotone sets. $f_n = \sum_{i=1}^n \alpha_i 1_{A_i}$ is a series of functions monotone pointwise converging to f. $F = \int f d\psi_{X\times Y}$ and $F^d = \int\left(\int f d\psi_X\right) d\psi_Y$. F_n and F_n^d denote the analogously defined integrals of f_n. The above section shows $F_n = F_n^d$. $f_1 \leq f_n \leq f$ implies with the General Bounded Convergence theorem (see Denneberg (1994, p. 101))

$$F_n^d \nearrow_{n\to\infty} F^d. \tag{27}$$

$f_n \leq f$ implies $F_n \leq F$ and $F_n^d \leq F^d$. Further $F \leq F^d$ holds. Taken together yields $F_n^d \leq F \leq F^d$ and with (27) one gets $F = F^d$. The proof for simple comonotone functions with an upper bound of 0 *follows identically. For any function f, the functions* $\max(0, f(x))$ *and* $\min(0, f(x))$ *are comonotone. Comonotone additivity, i.e.* $\mathbb{CE}(f) = \mathbb{CE}\left[\max(0, f(x))\right] + \mathbb{CE}\left[\min(0, f(x))\right]$, *of the Choquet integral together with the Fubini theorem for simple comonotone functions with an upper or lower bound completes the proof for the class of simple comonotone functions. This expression is well defined due to the integrability condition for f. For an arbitrary, comonotone, measurable function f*

$$f_p = c_p(f) \text{ with } c_p(x) = \sup(\frac{k}{2^p}, k \in \mathbb{Z}, \frac{k}{2^p} \leq x) \tag{28}$$

defines a sequence of simple comonotone functions $\{f_p\}_{p\in\mathbb{N}}$. Then

$$f_p(x, y) \leq f(x, y) \leq f_p(x, y) + \frac{1}{2^p} \ \forall x, y \tag{29}$$

follows. $F = \int f d\psi_{X\times Y}$ and $F^d = \int \left(\int f d\psi_X\right) d\psi_Y$. F_p and F_p^d denote the analogously defined integrals of f_p. Then monotonicity and additivity of a constant imply

$$F_p \leq F \leq F_p + \frac{1}{2^p} \text{ and} \tag{30a}$$

$$F_p^d \leq F^d \leq F_p^d + \frac{1}{2^p} \ \forall p \in \mathbb{N}. \tag{30b}$$

Since $\{f_p\}_{p\in\mathbb{N}}$ are simple comonotone functions the proof of the former part yields $F_p = F_p^d$. Taking the limit $p \to \infty$ yields

$$F = F^d. \tag{31}$$

Proof 5 *Proof of corollary 3*

For arbitrary (not only for comonotone) $S \in \Sigma_{X\times Y}$

$$\begin{aligned}\left(\psi_X \overset{M}{\otimes} \pi\right)(S) &= \int_{\Sigma_Y}\left(\int_{\Sigma_X} u_{T_X\times T_Y}(S)d\mu^{\psi_X}\right)d\mu^{\pi} \\ &= \int_{\Omega_Y}\left(\int_{\Sigma_X} u_{T_X\times \{y\}}(S)d\mu^{\psi_X}\right)d\mu^{\pi} \\ &= \int_{\Omega_Y}\underbrace{\left(\int_{\Sigma_X} u_{T_X}(S_y)d\mu^{\psi_X}\right)}_{=\psi_X(S_y)}d\mu^{\pi} \\ &= \int_{\Omega_Y}\left(\int_{\Omega_X} 1_S d\psi_X\right)d\pi\end{aligned}$$

holds where $S_y = \{x : (x, y) \in S\}$ *is the* y*-section of* S*. The second equation yields because* μ^π *is the natural injection of an additive probability* π *into the larger space* Σ_Y*. Thus integrating with respect to* π *is equal to integrating with respect to* μ^π*. Further for* $f \geq 0$

$$\begin{aligned}\int f d\left(\psi_X \overset{M}{\otimes} \pi\right) &= \int_0^\infty \left(\psi_X \overset{M}{\otimes} \pi\right)(f \geq \alpha)d\alpha \\ &= \int_0^\infty \left(\int_{\Omega_Y}\left(\int_{\Omega_X} 1_{(f\geq\alpha)}d\psi_X\right)d\pi\right)d\alpha\end{aligned}$$

holds. For σ*-additive measures* α *and* π *Fubini may be applied (i.e.* $d\alpha d\pi = d\pi d\alpha$*)*

$$\begin{aligned}\ldots &= \int_{\Omega_Y}\left(\underbrace{\int_0^\infty\left(\int_{\Omega_X} 1_{(f\geq\alpha)}d\psi_X\right)d\alpha}_{=\int f d\psi_X}\right)d\pi \\ &= \int\left(\int f d\psi_X\right)d\pi\end{aligned}$$

The extension for negative f *follows analogously.*

Proof 6 *Proof of corollary 4*

Proof by induction:

1. $n = 1$*:* $\int 1_B du_A \overset{M}{\otimes} \psi_Y = u_A \overset{M}{\otimes} \psi_Y(B) = \psi_Y(\{y : (A, y) \subseteq B\}) = \int \left(\int 1_B du_A\right) d\psi_Y$

2. $n \to n + 1$*: For* $i < j \implies A_i \subseteq A_j$ *are* $\{y : (A_i, y) \subseteq B\}$ *and* $\{y : (A_j, y) \subseteq B\}$ *comonotone. Therefore* $\int \left(\int 1_B d\sum_{i=1}^n \lambda_i u_{A_i}\right) d\psi_Y = \sum_{i=1}^n \lambda_i \int \left(\int 1_B du_{A_i}\right) d\psi_Y$ *due to comonotone additivity of the Choquet integral.*

Proof 7 *Proof of corollary 5*

One needs to show that $\psi \in \mathcal{F}(\psi_X, \psi_Y) \Rightarrow \overline{\psi} \in \mathcal{F}\left(\overline{\psi_X}, \overline{\psi_Y}\right)$.

$$\begin{aligned}
\overline{\psi}(C) &= 1 - \psi(C^c) \\
&= 1 - \int \left(\int 1_{C^c} d\psi_X \right) d\psi_Y \\
&= \int 1 - \left(\int 1_{C^c} d\psi_X \right) d\overline{\psi_Y} \\
&= \int \left(\int 1 - 1_{C^c} d\overline{\psi_X} \right) d\overline{\psi_Y} \\
&= \int \left(\int 1_C d\overline{\psi_X} \right) d\overline{\psi_Y}
\end{aligned}$$

Thus $\mathcal{F}(\psi_X, \psi_Y)^D \subseteq \mathcal{F}\left(\overline{\psi_X}, \overline{\psi_Y}\right)$. *Using this relation twice yields* $\mathcal{F}\left(\overline{\psi_X}, \overline{\psi_Y}\right)^D \subseteq \mathcal{F}\left(\overline{\overline{\psi_X}}, \overline{\overline{\psi_Y}}\right)$ *or* $\mathcal{F}(\psi_X, \psi_Y) = \mathcal{F}(\psi_X, \psi_Y)^{DD} \subseteq \mathcal{F}\left(\overline{\psi_X}, \overline{\psi_Y}\right)^D \subseteq \mathcal{F}\left(\overline{\overline{\psi_X}}, \overline{\overline{\psi_Y}}\right) = \mathcal{F}(\psi_X, \psi_Y)$.

Proof 8 *Proof of corollary 6*

$\mathcal{F}(u_A, u_B) = \left\{ u_A \overset{M}{\otimes} u_B \right\} \Rightarrow \forall \psi \in \mathcal{F}(\overline{u_A}, \overline{u_B}) : \overline{\psi} = u_A \overset{M}{\otimes} u_B$ *and* $\overline{\overline{\psi}} = \psi$ *completes the proof.*